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MAPS FROM THE ENVELOPING ALGEBRA OF THE POSITIVE WITT ALGEBRA TO REGULAR ALGEBRAS

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ABSTRACT. We construct homomorphisms from the universal enveloping algebra of the positive (part of the) Witt algebra to several different Artin-Schelter regular algebras, and determine their kernels and images. As a result, we produce elementary proofs that the universal enveloping algebras of the Virasoro algebra, the Witt algebra, and the positive Witt algebra are neither left nor right noetherian.

0. INTRODUCTION

Let $\Bbbk$ be a field of characteristic 0. All vector spaces, algebras, $\otimes$ are over $\Bbbk$, unless stated otherwise. In this work, we construct and study homomorphisms from the universal enveloping algebra of the positive part of the Witt algebra to *Artin-Schelter (AS-)regular algebras*. The latter serve as homological analogues of commutative polynomial rings in the field of noncommutative algebraic geometry.

To begin, consider the Lie algebras below.

Definition 0.1 (V, W, W_+)**.** We define the following Lie algebras:

(a) The *Virasoro algebra* is defined to be the Lie algebra V with basis $\{e_n\}_{n\in\mathbb{Z}} \cup \{c\}$ and Lie bracket $[e_n, c] = 0$, $[e_n, e_m] = (m-n)e_{n+m} + \frac{c}{12}(m^3 - m)\delta_{n+m,0}$.

(b) The *Witt* (or *centerless Virasoro*) *algebra* is defined to be the Lie algebra W with basis $\{e_n\}_{n\in\mathbb{Z}}$ and Lie bracket $[e_n, e_m] = (m-n)e_{n+m}$.

(c) The *positive* (*part of the*) *Witt algebra* is defined to be the Lie subalgebra W_+ of W generated by $\{e_n\}_{n\geq 1}$.

For any Lie algebra $\mathfrak{g}$, we denote its universal enveloping algebra by $U(\mathfrak{g})$.

Further, consider the following algebras.

Notation 0.2 (S, R)**.** Let S be the algebra generated by u, v, w, subject to the relations

$$uv - vu - v^2 \;=\; uw - wu - vw \;=\; vw - wv \;=\; 0.$$

Let R be the *Jordan plane*, generated by u, v, subject to the relation $uv - vu - v^2 = 0$.

It is well-known that R is an AS-regular algebra of global dimension 2. Moreover, we see by Lemma 1.3 that S is also AS-regular, of global dimension 3.

This work focuses on maps that we construct from the enveloping algebra $U(W_+)$ to both R and S, given as follows:

Definition 0.3 (ϕ, λ_a)**.** Let $\phi : U(W_+) \to S$ be the algebra homomorphism induced by defining

$$\phi(e_n) = (u - (n-1)w)v^{n-1}. \tag{0.4}$$

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For $a \in \Bbbk$, let $\lambda_a : U(W_+) \to R$ be the algebra homomorphism induced by defining

$$\lambda_a(e_n) = (u - (n-1)av)v^{n-1}. \tag{0.5}$$

That ϕ and λ_a are well-defined is Lemma 1.5.

Our main result is that we understand fully the kernels and images of the maps above, as presented below.

Theorem 0.6. *We have the following statements about the kernels and images of the maps ϕ and λ_a.*

(a) [Propositions 2.5, 2.8] $\ker \lambda_a$ *is equal to the ideal* $(e_1e_3 - e_2^2 - e_4)$ *if* $a = 0, 1$; *or is an ideal generated by one element of degree 5 and two elements of degree 6 (listed in Proposition 2.8) if* $a \neq 0, 1$.

(b) [Proposition 2.1] $\lambda_a(U(W_+))$ *is equal to* $\Bbbk + uR$ *if* $a = 0$; *is equal to* $\Bbbk + Ru$ *if* $a = 1$; *or contains* $R_{\geq 4}$ *if* $a \neq 0, 1$. *For all* a, *the image of* λ_a *is noetherian.*

(c) [Theorem 5.1] $\ker \phi$ *is equal to* $(e_1e_5 - 4e_2e_4 + 3e_3^2 + 2e_6)$.

The image of ϕ will be discussed later in the introduction, after Theorem 0.10.

The result above has a surprising application. In [SW14, Theorem 0.5 and Corollary 0.6], the authors established that $U(W_+)$, $U(W)$, $U(V)$ are neither left nor right noetherian through relatively indirect means, using the techniques of [Sie11]. In particular, we were not able to give an example of a non-finitely-generated right or left ideal in any of these enveloping algebras. However, in the course of proving Theorem 0.6, we produce an elementary and constructive proof of [SW14, Theorem 0.5 and Corollary 0.6]. Namely, we obtain:

Theorem 0.7 (Proposition 2.5, Theorem 3.3)**.** *The ideal* $\ker \lambda_0 = \ker \lambda_1 = (e_1e_3 - e_2^2 - e_4)$ *is not finitely generated as either a left or a right ideal of* $U(W_+)$.

We prove this theorem by noting that λ_0 factors through ϕ, and by studying $B := \phi(U(W_+))$. A key step is to compute $I := \phi(\ker \lambda_0)$, and to show that I is not finitely generated as a left or right ideal of B.

Note that the map (0.5) can be extended to W to define a map which we denote by

$$\widehat{\lambda}_a : U(W) \to R[v^{-1}].$$

We also have:

Theorem 0.8 ((3.10), Theorem 3.12)**.** *The ideal* $\ker \widehat{\lambda}_0 = \ker \widehat{\lambda}_1$ *is not finitely generated as either a left or right ideal of* $U(W)$.

We remark that $R[v^{-1}]$ is isomorphic to the ring $\Bbbk[x, x^{-1}, \partial]$, which is a familiar localization of the Weyl algebra. To see this, set $v = x$ and $u = x^2\partial$, so $\partial x = x\partial + 1$. Then, $uv - vu = x^2 = v^2$. We obtain that

$$\widehat{\lambda}_1(e_n) = v^{n-1}u = x^{n+1}\partial.$$

Thus, $\widehat{\lambda}_1$ is a well-known homomorphism.

We now compare Theorem 0.7 with our earlier proof (in [SW14]) that $U(W_+)$ is not left or right noetherian. The earlier proof used a ring homomorphism ρ with a more complicated definition:

Notation 0.9 $(X,\ f,\ \tau,\ \rho)$**.** Take $\mathbb{P}^3 := \mathbb{P}^3_{\Bbbk}$ with coordinates w, x, y, z. Let $X = V(xz - y^2) \subseteq \mathbb{P}^3$ be the projective cone over a smooth conic in $\mathbb{P}^2$.

Define an automorphism τ of X by

$$\tau([w : x : y : z]) = [w - 2x + 2z : z : -y - 2z : x + 4y + 4z].$$

Denote the pullback of τ on $\Bbbk(X)$ by τ^*, so that $g^\tau := \tau^* g = g \circ \tau$ for $g \in \Bbbk(X)$. Form the ring $\Bbbk(X)[t; \tau^*]$ with multiplication $tg = g^\tau t$ for all $g \in \Bbbk(X)$. Let

$$f := \frac{w + 12x + 22y + 8z}{12x + 6y},$$

considered as a rational function in $\Bbbk(X)$. Now let $\rho : U(W_+) \to \Bbbk(X)[t;\tau^*]$ be the algebra homomorphism induced by setting $\rho(e_1) = t$ and $\rho(e_2) = ft^2$.

That ρ is well-defined is [SW14, Proposition 1.5].

The method [SW14] made a number of reductions to show that $\rho(U(W_+))$ is not left or right noetherian. That proof can now be streamlined via the next result.

Theorem 0.10 (Theorem 4.1)**.** *We have that* $\ker\rho = \ker\phi = \bigcap_{a\in\Bbbk}\ker\lambda_a$.

Since we show that $\phi(U(W_+))$ is not left or right noetherian in the course of proving Theorem 0.7, we have by Theorems 0.6(c) and 0.10 that $\rho(U(W_+)) \cong \phi(U(W_+)) \cong U(W_+)/(e_1e_5 - 4e_2e_4 + 3e_3^2 + 2e_6)$ is neither left or right noetherian.

We end by discussing an open question that was first brought to our attention by Lance Small.

Question 0.11. Does $U(W_+)$ satisfy the ascending chain condition on *two-sided* ideals?

Our result here is only partial: we show that

Proposition 0.12 (Proposition 6.6)**.** *The ring* $B := \phi(U(W_+))$ *satisfies the ascending chain condition on two-sided ideals.*

Of course, this yields no direct information on the question for $U(W_+)$.

We have the following conventions throughout the paper. We take $\mathbb{N} = \mathbb{Z}_{\geq 0}$ to be the set of non-negative integers. If r is an element of a ring A, then (r) denotes the two-sided ideal ArA generated by r. If $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is a graded $\Bbbk$-algebra (or graded module), then we define the Hilbert series

$$\operatorname{hilb} A = \textstyle\sum_{n\in\mathbb{Z}} \dim_\Bbbk A_n t^n.$$

This article is organized as follows. We present preliminary results in Section 1, including an alternative way of multiplying elements in S and in R (Lemma 1.3); this method will be employed throughout, sometimes without mention. In Section 2, we discuss the maps λ_a and prove parts (a,b) of Theorem 0.6. In Section 3 we use the map λ_0 to establish Theorem 0.7; we also prove Theorem 0.8.

Before proceeding to study the map ϕ, we present its connection with the map ρ, the key homomorphism of [SW14]. Namely, in Section 4, we establish Theorem 0.10. Then in Section 5, we verify part (c) of Theorem 0.6. Our last result, Proposition 0.12, is presented in Section 6. Proofs of computational claims via Maple and Macaulay2 routines and a known result in ring theory to which we could not find a reference are provided in the appendix.

1. Preliminaries

The main focus of this paper is the universal enveloping algebra of the positive Witt algebra, W_+. To begin, we recall some basic facts about the algebra $U(W_+)$.

Lemma 1.1. *Recall Definition 0.1(c).*

(a) *We have the following isomorphism:*

$$U(W_+) \cong \frac{\Bbbk\langle e_1, e_2\rangle}{\begin{pmatrix} [e_1,[e_1,[e_1,e_2]]] + 6[e_2,[e_2,e_1]], \\ [e_1,[e_1,[e_1,[e_1,[e_1,e_2]]]]] + 40[e_2,[e_2,[e_2,e_1]]] \end{pmatrix}}.$$

(b) *The set* $\{e_{i_1}e_{i_2}\dots e_{i_k} \mid k\in\mathbb{N}$ *and* $1 \leq i_1 \leq i_2 \leq \dots \leq i_k \in \mathbb{N}\}$ *forms a* $\Bbbk$*-vector space basis of* $U(W_+)$.

Proof. Part (a) is [SW14, Lemma 1.1], and part (b) is clear from the definition of $U(W_+)$. □

Next, let us present some notation that we will use for the rest of the paper. We will work with the algebras R and S defined in Notation 0.2; note that we can view R as a subalgebra of S. In addition:

Notation 1.2 (Q)**.** Take Q to be the subalgebra of S generated by u, v, and vw.

In our first result, we provide an easy way to multiply elements in S. Recall from [Zha96] that a *Zhang twist* of a graded algebra L, by an automorphism μ of L, is the algebra L^μ, where $L^\mu = L$ as graded vector spaces and L^μ has multiplication $\ell * \ell' = \ell(\ell')^{\mu^i}$ for $\ell \in L_i$ and $\ell' \in L$.

Moreover, recall that an *Artin-Schelter* (*AS-*)*regular algebra* is a connected graded algebra A of finite global dimension, of finite injective dimension d with $\mathrm{Ext}^i_A({}_A\Bbbk, {}_AA) \cong \mathrm{Ext}^i_A(\Bbbk_A, A_A) \cong \delta_{i,d}\Bbbk$ (that is, A is *AS-Gorenstein*), and has finite Gelfand-Kirillov dimension. These algebras are important in noncommutative ring theory because they are noncommutative analogues of polynomial rings and share many of their good properties.

Lemma 1.3 (μ, ν)**.** *Let $\mu \in \mathrm{Aut}(\Bbbk[x,y,z])$ be defined by*

$$\mu(x) = x - y, \quad \mu(y) = y, \quad \mu(z) = z.$$

Then, S is isomorphic to the Zhang twist $\Bbbk[x,y,z]^\mu$. Further, μ restricts to an automorphism of $\Bbbk[x,y,yz]$, which we also denote by μ, and to an automorphism ν of $\Bbbk[x,y]$. We have that $R \cong \Bbbk[x,y]^\nu$ and $Q \cong \Bbbk[x,y,yz]^\mu$ as graded $\Bbbk$-algebras. As a consequence, S, R, and Q are AS-regular algebras.

Proof. To see that $S \cong \Bbbk[x,y,z]^\mu$, we emphasize that

$$\begin{gathered}\text{the variables } u,v,w \text{ of } S \text{ have noncommutative multiplication,}\\ \text{the variables } x,y,z \text{ of } \Bbbk[x,y,z] \text{ have commutative multiplication, and}\\ * \text{ denotes the noncommutative multiplication on } \Bbbk[x,y,z]^\mu \text{ defined by}\\ \ell * \ell' = \ell(\ell')^{\mu^i} \text{ for } \ell \in \Bbbk[x,y,z]_i \text{ and } \ell' \in \Bbbk[x,y,z].\end{gathered} \tag{1.4}$$

Now,

$$\begin{array}{lllllll} y*x & = yx^\mu & = y(x-y) & = (x-y)y & = xy - y^2 & = xy^\mu - yy^\mu & = x*y - y*y, \\ z*x & = zx^\mu & = z(x-y) & = (x-y)z & = xz - yz & = xz^\mu - yz^\mu & = x*z - y*z, \\ z*y & = zy^\mu & = zy & = yz & = yz^\mu & & = y*z. \end{array}$$

Thus, if we identify u,v,w with x,y,z, respectively, then the relations of S hold in $\Bbbk[x,y,z]^\mu$, and $S \cong \Bbbk[x,y,z]^\mu$ as graded $\Bbbk$-algebras.

That μ restricts to automorphisms of $\Bbbk[x,y]$ and $\Bbbk[x,y,yz]$ is immediate, and the other isomorphisms hold by a similar argument. Moreover, the last statement follows as commutative polynomial rings are AS-regular and this property is preserved under Zhang twist by [Zha96, Theorem 1.3(i)]. □

Now we verify that the algebra homomorphisms λ_a and ϕ from Definition 0.3 are well-defined.

Lemma 1.5. *The maps ϕ and λ_a of Definition 0.3 are well-defined homomorphisms of graded $\Bbbk$-algebras.*

Proof. We check that ϕ respects the Witt relations given in Definition 0.1(b), by using Lemma 1.3 and (1.4):

$$\begin{aligned} \phi(e_ne_m - e_me_n) &= (u-(n-1)w)v^{n-1}(u-(m-1)w)v^{m-1} - (u-(m-1)w)v^{m-1}(u-(n-1)w)v^{n-1} \\ &= (x-(n-1)z)(x-(m-1)z)^{\mu^n}y^{n+m-2} - (x-(m-1)z)(x-(n-1)z)^{\mu^m}y^{n+m-2} \\ &= [(x-(n-1)z)(x-ny-(m-1)z) - (x-(m-1)z)(x-my-(n-1)z)]y^{n+m-2} \\ &= (m-n)xy^{n+m-1} + (n(n-1)-m(m-1))y^{n+m-1}z \\ &= (m-n)(x-(n+m-1)z)y^{n+m-1} \\ &= (m-n)(u-(n+m-1)w)v^{n+m-1} \\ &= (m-n)\phi(e_{n+m}). \end{aligned}$$

So, the claim holds for ϕ.

Similarly, we verify that λ_a respects the Witt relations:

$$\begin{aligned}\lambda_a(e_ne_m - e_me_n) &= (u-(n-1)av)v^{n-1}(u-(m-1)av)v^{m-1} - (u-(m-1)av)v^{m-1}(u-(n-1)av)v^{n-1}\\ &= [(x-(n-1)ay)(x-ny-(m-1)ay) - (x-(m-1)ay)(x-my-(n-1)ay)]y^{n+m-2}\\ &= (m-n)(x-a(n+m-1)y)y^{n+m-1}\\ &= (m-n)(u-a(n+m-1)v)v^{n+m-1}\\ &= (m-n)\lambda_a(e_{n+m}).\end{aligned}$$

Thus, the claim holds for λ_a. □

Next, we define the key algebras $A(a)$ and B that we will use throughout.

Notation 1.6 ($A(a)$, B)**.** Take $a \in \Bbbk$ and let $A(a)$ denote the subalgebra $\lambda_a(U(W_+))$ of R. Further, let B denote the subalgebra $\phi(U(W_+))$ of S.

We point out a useful observation.

Lemma 1.7. *We have that $B \subseteq Q$.*

Proof. We get that $\phi(e_1) = u$ and $\phi(e_2) = (u-w)v = uv - vw$ are in Q. By Lemma 1.1(a), B is generated by these elements, so we are done. □

2. The kernel and image of the maps λ_a

The goal of this section is to analyze the maps λ_a from Definition 0.3, which are well-defined by Lemma 1.5. In particular, we verify Theorem 0.6(a,b).

To proceed, recall Notations 0.2 and 1.6. We first compute the factor rings $A(a)$, proving Theorem 0.6(b).

Proposition 2.1. *We have that $A(0) = \Bbbk + uR$ (a right idealizer in R), that $A(1) = \Bbbk + Ru$ (a left idealizer in R), and that $A(a)_{\geq 4} = R_{\geq 4}$ if $a \neq 0, 1$. For all a, the ring $A(a)$ is noetherian.*

Proof. Recall from Lemma 1.1(a) that $U(W_+)$ is generated by e_1 and e_2. We have that $\lambda_0(e_1) = u$ and $\lambda_0(e_2) = uv$. These elements generate $\Bbbk + uR$. Moreover, $\lambda_1(e_1) = u$ and $\lambda_1(e_2) = (u-v)v = vu$, and these elements generate $\Bbbk + Ru$. That the rings $A(0)$ and $A(1)$ are noetherian follows from [SZ94, Lemma 2.2(iii) and Theorem 2.3(i.a)].

When $a \neq 0, 1$, we must show that $R_{\geq 4} \subseteq A(a)$. Since $uR_n + R_nu = R_{n+1}$ for $n \geq 1$ and since $\dim_\Bbbk R_4 = 5$, the proof boils down to showing that the set of elements

$$\lambda_a(e_1^4), \quad \lambda_a(e_1^2e_2), \quad \lambda_a(e_1e_2e_1), \quad \lambda_a(e_2e_1^2), \quad \lambda_a(e_2^2),$$

is $\Bbbk$-linearly independent, for $a \neq 0, 1$. Using Lemma 1.3 and (1.4), consider the following calculations,

$$\begin{array}{lllll}\lambda_a(e_1^4) & = u^4 & = xx^\mu x^{\mu^2}x^{\mu^3} & = x(x-y)(x-2y)(x-3y) & =: r_1,\\ \lambda_a(e_1^2e_2) & = u^2(u-av)v & = xx^\mu(x-ay)^{\mu^2}y^{\mu^3} & = x(x-y)(x-(2+a)y)y & =: r_2,\\ \lambda_a(e_1e_2e_1) & = u(u-av)vu & = x(x-ay)^\mu y^{\mu^2}x^{\mu^3} & = x(x-(1+a)y)y(x-3y) & =: r_3,\\ \lambda_a(e_2e_1^2) & = (u-av)vu^2 & = (x-ay)y^\mu x^{\mu^2}x^{\mu^3} & = (x-ay)y(x-2y)(x-3y) & =: r_4,\\ \lambda_a(e_2^2) & = (u-av)v(u-av)v & = (x-ay)y^\mu(x-ay)^{\mu^2}y^{\mu^3} & = (x-ay)y(x-(2+a)y)y & =: r_5.\end{array}$$

By direct computation, we see that $r_1, \ldots, r_5$ are linearly independent if $a \neq 0, 1$.

Further, since $A(a)$ and R are equal in large degree and R is noetherian, $A(a)$ is noetherian by [Sta85, Lemma 1.4]. □

Our next goal is to compute the kernels of the maps λ_a and establish Theorem 0.6(a). We will use the following notation:

Notation 2.2 (π, π_a, π_B)**.** Let $\Bbbk\langle t_1, t_2\rangle$ be the free algebra, which we grade by setting $\deg t_i = i$. We set the notation below:

- $\pi : \Bbbk\langle t_1, t_2\rangle \to U(W_+)$ is the algebra map given by $\pi(t_1) = e_1$ and $\pi(t_2) = e_2$;
- $\pi_a : \Bbbk\langle t_1, t_2\rangle \to R$ is the algebra map given by $\pi_a(t_1) = \lambda_a(e_1) = u$ and $\pi_a(t_2) = \lambda_a(e_2) = (u - av)v$, for $a \in \Bbbk$. The image of π_a is $A(a)$. Note that $\pi_a = \lambda_a \circ \pi$.
- $\pi_B : \Bbbk\langle t_1, t_2\rangle \to S$ is the algebra map given by $\pi_B(t_1) = \phi(e_1) = u$ and $\pi_B(t_2) = \phi(e_2) = uv - vw$. The image of π_B is B. Note that $\pi_B = \phi \circ \pi$.

In the next result, we compute a presentation of the algebra $A(0)$.

Lemma 2.3. *The kernel of π_0 is generated by*

$$\begin{aligned} q &:= t_1^2t_2 - t_2t_1^2 - 2t_2^2, \\ q' &:= t_1^3t_2 - 3t_1^2t_2t_1 + 3t_1t_2t_1^2 - t_2t_1^3 + 6t_2^2t_1 - 12t_2t_1t_2 + 6t_1t_2^2 \end{aligned}$$

as a two-sided ideal.

Proof. Let $A = A(0)$, and consider the exact sequence of right A-modules:

$$0 \longrightarrow K \longrightarrow A[-1] \oplus A[-2] \xrightarrow{(u,uv)} A \longrightarrow \Bbbk \longrightarrow 0.$$

We make the following claim:

Claim. As a right A-module, K is generated by $(u^2v, -u(u + 2v))$ and $(u^2v^2, -u(u + 2v)v)$.

Assume the claim. It is well-known that one may deduce generators and relations of a connected graded $\Bbbk$-algebra from the first few terms in a minimal resolution of the trivial module $\Bbbk$. The precise method is given in Proposition 7.1 in the appendix. Using the notation of that result: take $b_1^1 = u^2v$, $b_2^1 = -u(u+2v)$, $b_1^2 = u^2v^2$, and $b_2^2 = -u(u + 2v)v$. Moreover, take $\widetilde{b}_1^1 = t_1t_2$, $\widetilde{b}_2^1 = -t_1^2 - 2t_2$, $\widetilde{b}_1^2 = t_1^2t_2 - t_1t_2t_1$, and $\widetilde{b}_2^2 = 2t_2t_1 - 3t_1t_2$. Note that $\pi_0(\widetilde{b}_j^i) = b_j^i$ for $i, j = 1, 2$. Now we obtain by Proposition 7.1 that

$$\begin{aligned} q_1 &:= t_1(\widetilde{b}_1^1) + t_2(\widetilde{b}_2^1) &&= t_1^2t_2 - t_2t_1^2 - 2t_2^2 \\ q_2 &:= t_1(\widetilde{b}_1^2) + t_2(\widetilde{b}_2^2) &&= t_1^3t_2 - t_1^2t_2t_1 + 2t_2^2t_1 - 3t_2t_1t_2 \end{aligned}$$

generate $\ker \pi_0$. Observe that $q = q_1$ and that

$$q' - 4q_2 \;=\; -3t_1^3t_2 + t_1^2t_2t_1 + 3t_1t_2t_1^2 - t_2t_1^3 - 2t_2^2t_1 + 6t_1t_2^2 \;=\; -3t_1q + qt_1 \;\in (q).$$

Thus, $\ker \pi_0$ is generated by q and q', as desired.

So it remains to prove the claim.

Proof of Claim. Note that there is an isomorphism of graded right A-modules $\beta : uA \cap uvA \to K$ given by $\beta(r) = (u^{-1}r, -(uv)^{-1}r)$.

Take $M := A \cap vA$. Since $A = \Bbbk + uR$, it is easy to show that $M = uR \cap vuR$, and in particular, is a right R-module. Since $(uR+vuR)_{\geq 2} = R_{\geq 2}$, we get that $\dim_\Bbbk M_n = \dim_\Bbbk R_{n-1} + \dim_\Bbbk R_{n-2} - \dim_\Bbbk R_n = n-2$ for $n \geq 2$, and $\dim_\Bbbk M_n = 0$ for $n < 2$. Moreover, $u^2v = vu(u+2v) \in M$, so $u^2vR \subseteq M$ and $\mathrm{hilb}(u^2vR) = \mathrm{hilb}\, M$. So, $M = u^2vR$. Now

$$uA \cap uvA = uM \;=\; u^3vR \overset{(*)}{=} u^3vA + u^3v^2A \;=\; uvu(u + 2v)A + uvu(u + 2v)vA,$$

where the equality $(*)$ holds as $R = A + vA$. Apply the map β to the right-hand-side of the equation above to yield the desired result. $\square$

We can now understand $\ker \lambda_0$ and $\ker \lambda_1$. We first prove that:

Lemma 2.4. *We have* $\ker \lambda_0 = \ker \lambda_1$.

Proof. Working in the quotient division ring of R, we have: $u^{-1}\lambda_0(e_n)u = v^{n-1}u = \lambda_1(e_n)$. So for any $f \in U(W_+)$, we have $\lambda_1(f) = u^{-1}\lambda_0(f)u$. The result follows. $\square$

Proposition 2.5. *We have that* $\ker \lambda_a = (e_1e_3 - e_2^2 - e_4)$ *for* $a = 0, 1$.

Proof. We first check that $e_1e_3 - e_2^2 - e_4$ is indeed in $\ker \lambda_0$ as follows:

$$\lambda_0(e_1e_3 - e_2^2 - e_4) = u(uv^2) - (uv)(uv) - uv^3 = \; u^2v^2 - u(uv - v^2)v - uv^3 \; = \; 0.$$

Recall that $\pi_0 = \lambda_0 \circ \pi$. So, Lemma 2.3 implies that $\ker \lambda_0 = \pi(\ker \pi_0)$ is generated by elements $\pi(q)$ and $\pi(q')$ in $U(W_+)$. Now $\pi(q') = 0$ by Lemma 1.1(a), so $\ker \lambda_0$ is generated by $\pi(q)$. Moreover,

$$\pi(q) = e_1^2e_2 - e_2e_1^2 - 2e_2^2 = 2\left(e_1(e_1e_2 - e_2e_1) - e_2^2 - \left(\frac{1}{2}e_1^2e_2 - e_1e_2e_1 + \frac{1}{2}e_2e_1^2\right)\right) = 2(e_1e_3 - e_2^2 - e_4),$$

using the relation $[e_n, e_m] = (m - n)e_{n+m}$ in $U(W_+)$. Thus, $\ker \lambda_0 = (e_1e_3 - e_2^2 - e_4)$, as claimed.

The result for $a = 1$ now follows by Lemma 2.4. $\square$

It remains to analyze $\ker \lambda_a$ with $a \neq 0, 1$. We do this in the next two results.

Lemma 2.6. *For* $a \neq 0, 1$*, the kernel of* λ_a *is generated in degrees 5 and 6.*

Proof. Take $A' := A(a)$. It suffices to show that the kernel of π_a is generated in degrees 5 and 6. Consider the exact sequence of right A'-modules:

$$0 \longrightarrow K \longrightarrow A'[-1] \oplus A'[-2] \xrightarrow{(u,(u-av)v)} A' \longrightarrow \Bbbk \longrightarrow 0.$$

We have that $uA' \cap (u - av)vA' \cong K$ as right A'-modules. As in the proof of Lemma 2.3, it now suffices to show that $uA' \cap (u - av)vA'$ is generated in degrees 5 and 6 as a right A'-module.

Let $J := uA' \cap (u - av)vA'$, and let $L := uR \cap (u - av)vR$. Note that $J \subseteq L$. Since $a \neq 0$, we get that $R_{\geq 2} = (uR + (u - av)vR)_{\geq 2}$. So, $\dim_\Bbbk L_n = \dim_\Bbbk R_{n-1} + \dim_\Bbbk R_{n-2} - \dim_\Bbbk R_n = n - 2$, for $n \geq 2$. So, $\dim_\Bbbk L_3 = 1$, and principally generated as a right R-module by an element of degree 3. In fact,

$$L = rR, \qquad \text{where } r := u(uv + (1 - a)v^2) = (uv - av^2)(u + 2v). \tag{2.7}$$

Since $A'_{\geq 4} = R_{\geq 4}$ by Proposition 2.1, we have $J_{\geq 6} = L_{\geq 6}$. By direct computation, one obtains that $J_i = 0$ for $i = 0, \ldots, 4$; one can also use Routine 7.2 in the appendix.

Let $J' = J_5A' + J_6A'$. We prove by induction that $J_n = J'_n$, for all $n \geq 5$. The statement is clear for $n = 5, 6$. For $n = 7$, we make the following assertion, the proof of which is presented in the appendix; see Claim 7.3.

Claim. We have that $J_5A'_2 \not\subseteq J_6A'_1$.

So for $n \geq 6$, we have $J_n = L_n = rR_{n-3}$. So $\dim_\Bbbk J_7 = 5$, and $\dim_\Bbbk J_6A'_1 = \dim_\Bbbk J_6 = 4$. With the claim, we obtain that $J_7 = J_5A'_2 + J_6A'_1$. Thus, $J_7 = J'_7$. Now for the induction step, suppose we have established that $J'_n = J_n$ and $J'_{n-1} = J_{n-1}$ for some $n \geq 7$. Then,

$$\begin{aligned} J_{n+1} \supseteq J'_{n+1} &= J'_n u + J'_{n-1}(u - av)v &&= J_n u + J_{n-1}(u - av)v \\ &= r(R_{n-3}u + R_{n-4}(u - av)v) &&= rR_{n-2} &&= J_{n+1}. \end{aligned}$$

The penultimate equality holds as $a \neq 1$. Thus, the lemma is verified. $\square$

Proposition 2.8. *If* $a \neq 0, 1$*, then* $\ker \lambda_a$ *is the ideal generated by the elements*

$$h_1 := e_1e_2^2 - e_1^2e_3 - (2a)e_2e_3 + (1 + 2a)e_1e_4 - (a^2 + a)e_5,$$

$$h_2 := e_1e_5 - 4e_2e_4 + 3e_3^2 + 2e_6$$

$$h_3 := -4e_1^2e_2^2 - 4e_2^3 + 4e_1^3e_3 + (20a^2 + 14a - 7)e_3^2 - (16a^2 + 18a + 5)e_1e_5 + (16a^3 + 36a^2 + 16a - 2)e_6.$$

Proof. By Lemma 2.6, we just need to produce linearly independent elements of $\ker \lambda_a$ in degrees 5 and 6. We have by Routine 7.2 that $\dim_{\Bbbk}(\ker \lambda_a)_5 = 1$ and that we can choose a basis of $(\ker \lambda_a)_5$ to be the element h_1. In fact, we verify that $\lambda_a(h_1) = 0$ using Lemma 1.3 and (1.4), while suppressing some μ superscripts:

$$\begin{aligned}
\lambda_a(h_1) \quad &= u(u-av)v(u-av)v - u^2(u-2av)v^2 - (2a)(u-av)v(u-2av)v^2 \\
&\quad +(1+2a)u(u-3av)v^3 - (a^2+a)(u-4av)v^4 \\
&= x(x-ay)^{\mu}y(x-ay)^{\mu^3}y - xx^{\mu}(x-2ay)^{\mu^2}y^2 - (2a)(x-ay)y(x-2ay)^{\mu^2}y^2 \\
&\quad +(1+2a)x(x-3ay)^{\mu}y^3 - (a^2+a)(x-4ay)y^4 \\
&= x(x-(1+a)y)y(x-(3+a)y)y - x(x-y)(x-(2+2a)y)y^2 - (2a)(x-ay)y(x-(2+2a)y)y^2 \\
&\quad +(1+2a)x(x-(1+3a)y)y^3 - (a^2+a)(x-4ay)y^4 \ = 0.
\end{aligned}$$

On the other hand, we have by Routine 7.2 that $\dim_{\Bbbk}(\ker \lambda_a)_6 = 4$ and that we can take a basis of $(\ker \lambda_a)_6$ to be h_2, h_3 along with

$$\begin{aligned}
h_4 &:= 4e_2^3 - 4e_1e_2e_3 + (7-4a)e_3^2 + (1+4a)e_1e_5 + (2-4a-4a^2)e_6, \\
h_5 &:= 4e_2^3 + (7-14a)e_3^2 - 4e_1^2e_4 + (5+14a)e_1e_5 + (2-16a-12a^2)e_6.
\end{aligned}$$

By direct computation we have:

$$\begin{aligned}
e_1h_1 &= e_1^2e_2^2 - e_1^3e_3 - (2a)e_1e_2e_3 + (1+2a)e_1^2e_4 - (a^2+a)e_1e_5, \\
h_1e_1 &= e_1e_2^2e_1 - e_1^2e_3e_1 - (2a)e_2e_3e_1 + (1+2a)e_1e_4e_1 - (a^2+a)e_5e_1, \\
&= e_1^2e_2^2 - e_1^3e_3 - (2+2a)e_1e_2e_3 + (2a)e_3^2 + (3+2a)e_1^2e_4 + (4a)e_2e_4 - (2+7a+a^2)e_1e_5 + 4(a^2+a)e_6.
\end{aligned}$$

Claim. We have that h_2, h_3, e_1h_1, h_1e_1 are $\Bbbk$-linearly independent and that

$$h_4 = 2a(2a+1)h_2 - h_3 - (6+4a)e_1h_1 + (2+4a)h_1e_1, \quad h_5 = 4a^2h_2 - h_3 - (4+4a)e_1h_1 + (4a)h_1e_1.$$

The proof is presented in the appendix; see Claim 7.5. Therefore, the result holds.

Now for the reader's convenience, we verify that $\lambda_a(h_i) = 0$ for $i = 2, 3$ using Lemma 1.3 and (1.4), while suppressing some μ superscripts:

$$\begin{aligned}
\lambda_a(h_2) \quad &= u(u-4av)v^4 - 4(u-av)v(u-3av)v^3 + 3(u-2av)v^2(u-2av)v^2 + 2(u-5av)v^5 \\
&= x(x-4ay)^{\mu}y^4 - 4(x-ay)y(x-3ay)^{\mu^2}y^3 + 3(x-2ay)y^2(x-2ay)^{\mu^3}y^2 + 2(x-5ay)y^5 \\
&= x(x-(1+4a)y)y^4 - 4(x-ay)y(x-(2+3a)y)y^3 + 3(x-2ay)y^2(x-(3+2a)y)y^2 + 2(x-5ay)y^5 \ = 0;
\end{aligned}$$

$$\begin{aligned}
\lambda_a(h_3) \quad &= -4u^2(u-av)v(u-av)v - 4(u-av)v(u-av)v(u-av)v + 4u^3(u-2av)v^2 \\
&\quad +(20a^2+14a-7)(u-2av)v^2(u-2av)v^2 - (16a^2+18a+5)u(u-4av)v^4 \\
&\quad +(16a^3+36a^2+16a-2)(u-5av)v^5 \\
&= -4xx^{\mu}(x-ay)^{\mu^2}y(x-ay)^{\mu^4}y - 4(x-ay)y(x-ay)^{\mu^2}y(x-ay)^{\mu^4}y + 4xx^{\mu}x^{\mu^2}(x-2ay)^{\mu^3}y^2 \\
&\quad +(20a^2+14a-7)(x-2ay)y^2(x-2ay)^{\mu^3}y^2 - (16a^2+18a+5)x(x-4ay)^{\mu}y^4 \\
&\quad +(16a^3+36a^2+16a-2)(x-5ay)y^5 \\
&= -4x(x-y)(x-(2+a)y)y(x-(4+a)y)y - 4(x-ay)y(x-(2+a)y)y(x-(4+a)y)y \\
&\quad +4x(x-y)(x-2y)(x-(3+2a)y)y^2 + (20a^2+14a-7)(x-2ay)y^2(x-(3+2a)y)y^2 \\
&\quad -(16a^2+18a+5)x(x-(1+4a)y)y^4 + (16a^3+36a^2+16a-2)(x-5ay)y^5 \ = 0.
\end{aligned}$$

$\square$

3. Elementary proofs that $U(W_+)$ and $U(W)$ are not noetherian

In this section, we establish the remaining part of Theorem 0.7, that $\ker\lambda_0 = \ker\lambda_1$ is not finitely generated as a left or right ideal of $U(W_+)$. We also prove Theorem 0.8.

We first focus on $U(W_+)$. Recall the map $\phi : U(W_+) \twoheadrightarrow B$ from Definition 0.3, and consider Notations 0.2, 1.2, 1.6, and 2.2 along with the following.

Notation 3.1 (p, I)**.** Let $p := \phi(e_1e_3 - e_2^2 - e_4)$ be an element of B, and let $I := (p)$ be a two-sided ideal of B. Note that by Proposition 2.5, $I = \phi(\ker\lambda_0) = \pi_B(\ker\pi_0)$.

We begin by establishing some basic facts about p and I.

Lemma 3.2. *We have the following statements:*

(a) $p = v^3w - v^2w^2$,
(b) *p is a normal element of S and of Q, and*
(c) $I = Qp$.

Proof. We employ Lemma 1.3 and (1.4) in all parts.

(a) Consider the computation in S below:

$$\begin{aligned} p = \phi(e_1e_3 - e_2^2 - e_4) &= u(u-2w)v^2 - (u-w)v(u-w)v - (u-3w)v^3 \\ &= x(x-2z)^{\mu}y^{\mu^2}y^{\mu^3} - (x-z)y^{\mu}(x-z)^{\mu^2}y^{\mu^3} - (x-3z)y^{\mu}y^{\mu^2}y^{\mu^3} \\ &= x(x-y-2z)y^2 - (x-z)y(x-2y-z)y - (x-3z)y^3 \\ &= y^3z - y^2z^2 \\ &= v^3w - v^2w^2. \end{aligned}$$

(b) From part (a), we get that p is a normal element of S, and of Q, since $vp = pv$, $wp = pw$, and

$$\begin{aligned} up &= u(v^3w - v^2w^2) & &= xy^{\mu}y^{\mu^2}y^{\mu^3}z^{\mu^4} - xy^{\mu}y^{\mu^2}z^{\mu^3}z^{\mu^4} \\ &= (y^3z - y^2z^2)x & &= (y^3z - y^2z^2)(x+4y)^{\mu^4} \\ &= (v^3w - v^2w^2)(u+4v) & &= p(u+4v). \end{aligned}$$

(c) On one hand, we get that $I = BpB \subseteq QpQ = Qp$, by Lemma 1.7 and part (b). On the other hand, recall that R is the subalgebra of Q generated by u, v. We will show by induction on i and j that $p(vw)^iR_{j-2i} \subseteq I$ for all $0 \le i \le \lfloor j/2 \rfloor$; this yields $pQ_j \subseteq I$.

The base case $i = j = 0$ holds since $p \in I$. For the induction step, assume that $p(vw)^iR_{j-2i} \subseteq I$. Now it suffices to show that (i) $p(vw)^iR_{j+1-2i} \subseteq I$, and (ii) $p(vw)^{i+1}R_{j-2i} \subseteq I$.

For (i), we have by induction that

$$I \ \supseteq \ up(vw)^iR_{j-2i} + p(vw)^iR_{j-2i}u =: I',$$

since u is a generator of B. Now consider the following computations, where we suppress the action of μ on invariant elements and on graded pieces of $\Bbbk[x,y]$:

$$\begin{aligned} I' &= x(y^3z - y^2z^2)(yz)^i\Bbbk[x,y]_{j-2i} + (y^3z - y^2z^2)(yz)^i\Bbbk[x,y]_{j-2i}x^{\mu^{j+4}} \\ &= (y^3z - y^2z^2)(yz)^ix\Bbbk[x,y]_{j-2i} + (y^3z - y^2z^2)(yz)^i(x+(j+4)y)\Bbbk[x,y]_{j-2i} \\ &= (y^3z - y^2z^2)(yz)^i\left[x\Bbbk[x,y]_{j-2i} + (x+(j+4)y)\Bbbk[x,y]_{j-2i}\right] \\ &= (y^3z - y^2z^2)(yz)^i\Bbbk[x,y]_{j+1-2i}, \end{aligned}$$

where the last equality holds since $j + 4 > 0$. Thus (i) holds.

For (ii), we get that $p(vw)^iR_{j+2-2i} \subseteq I$ by applying (i) twice. Now

$$I \ \supseteq \ p(vw)^iR_{j+2-2i} + p(vw)^iR_{j-2i}(uv - vw) \ \supseteq \ p(vw)^iR_{j-2i}(vw).$$

Note that $R_k(vw) = (vw)R_k$ for all k. So $I \ \supseteq \ p(vw)^{i+1}R_{j-2i}$ and we are done. $\square$

Now we complete the proof of Theorem 0.7.

Theorem 3.3. *The ideal I of B is not finitely generated as a left or right ideal. As a result, the kernel of λ_0 is not finitely generated as a left or right ideal of $U(W_+)$.*

Proof. Recall that $\ker\lambda_0 = (e_1e_3 - e_2^2 - e_4)$ by Proposition 2.5. It is clear that if $\ker\lambda_0$ is finitely generated as a left/right ideal of $U(W_+)$, then I is a finitely generated as a left/right ideal of B. Therefore, to show that $\ker\lambda_0$ is not finitely generated it suffices to show that ${}_BI$ and I_B are not finitely generated.

By way of contradiction, suppose that ${}_BI$ is finitely generated. Then, there exists $n \geq 4$ so that $BI_{\leq n} = I$. Since B is generated by u and $(u-w)v$, we get that

$$I_{n+1} \;=\; B_1I_n + B_2I_{n-1} \;=\; uI_n + (u-w)vI_{n-1}. \tag{3.4}$$

By Lemma 3.2, $I = Qp \subseteq SpS = Sp$. Since $vI \subseteq vSp \subseteq Sp$, we get by (3.4) that

$$I_{n+1} \subseteq uSp + (u-w)Sp = uSp + wSp. \tag{3.5}$$

By using Lemma 1.3 and (1.4), it is easy to see that $uS + wS = x\Bbbk[x,y,z] + z\Bbbk[x,y,z]$ and that a positive power of y cannot belong to the right-hand-side. So, a positive power of v cannot belong to $uS + wS$. Therefore,

$$v^{n-3}p \notin uSp + wSp. \tag{3.6}$$

On the other hand, $v^{n-3}p \in I_{n+1}$ by Lemma 3.2(c). This contradicts (3.5) and (3.6). Thus, ${}_BI$ is not finitely generated.

Next, suppose that I_B is finitely generated. Then, there exists $n \geq 4$ so that $I_{\leq n}B = I$, with

$$I_{n+1} \;=\; I_nB_1 + I_{n-1}B_2 \;=\; I_nu + I_{n-1}(u-w)v \;=\; I_nu + I_{n-1}v(u+v-w). \tag{3.7}$$

We get that $I, Iv \subseteq pS$ by Lemma 3.2(b). So, the right-hand-side of (3.7) is contained in $pSu + pS(v-w)$. With an argument similar to that in the previous paragraph, $Su + S(v-w)$ does not contain positive powers of v. So, $pv^{n-3} \notin I_nu + I_{n-1}v(u+v-w)$. On the other hand, $pv^{n-3} \in I_{n+1}$ by Lemma 3.2(b,c), which contradicts (3.7). Thus, I_B is not finitely generated. □

Remark 3.8. We do not know whether or not $\ker\lambda_a$ is finitely generated for $a \neq 0, 1$.

One can of course deduce from Theorem 3.3 that $U(W)$ and $U(V)$ are neither left nor right noetherian; see [SW14, Lemma 1.7]. Nevertheless, a direct proof that $U(W)$ is not left or right noetherian is of independent interest, and we give such a result to end the section. First, we establish some notation.

Notation 3.9 ($\widehat{S}$, $\widehat{R}$, $\widehat{B}$, $\widehat{\phi}$, $\widehat{\lambda}_a$, η_a, $\widehat{I}$)**.** Since v is normal in S and in R, we may invert it. Let $\widehat{S} := S[v^{-1}]$, and let $\widehat{R} := R[v^{-1}]$.

Note that ϕ extends to an algebra homomorphism $\widehat{\phi} : U(W) \to \widehat{S}$ defined by (0.4) for all $n \in \mathbb{Z}$. Likewise, λ_a extends to an algebra homomorphism $\widehat{\lambda}_a : U(W) \to \widehat{R}$ defined by (0.5) for all $n \in \mathbb{Z}$. For $a \in \Bbbk$ define $\eta_a : \widehat{S} \to \widehat{R}$ by $u \mapsto u, v \mapsto v, w \mapsto av$. Note that $\widehat{\lambda}_a = \eta_a\widehat{\phi}$.

Let $\widehat{B} := \widehat{\phi}(U(W))$. Finally, let $\widehat{I} = \widehat{\phi}(\ker\widehat{\lambda}_0)$. Note that $\widehat{I} = \widehat{B} \cap \ker\eta_0$.

We first note that the proof of Lemma 2.4 extends to $U(W)$ to give

$$\ker\widehat{\lambda}_0 = \ker\widehat{\lambda}_1. \tag{3.10}$$

We now show:

Proposition 3.11. *Recall $p = \phi(e_1e_3 - e_2^2 - e_4) = w(v-w)v^2$ from Notation 3.1 and Lemma 3.2. We have that*

$$\widehat{I} = \widehat{B} \cap \ker\eta_0 = \widehat{B} \cap \ker\eta_1 = \widehat{B}p\widehat{B} = \widehat{S}p = p\widehat{S}.$$

Proof. We first show that $\widehat{B}p\widehat{B} = \widehat{S}p = p\widehat{S}$. Certainly, $\widehat{B}p\widehat{B} \subseteq \widehat{S}p\widehat{S} = \widehat{S}p = p\widehat{S}$, where the last two equalities hold because a normal element of S will also be normal in $\widehat{S}$.

For the other direction, we will show that $\widehat{R}w^jp \subseteq \widehat{B}p\widehat{B}$ for all $j \geq 0$ by induction. Since $\widehat{S} = \widehat{R} \cdot \Bbbk[w]$, this will imply that $\widehat{S}p \subseteq \widehat{B}p\widehat{B}$. So assume that $w^jp \in \widehat{B}p\widehat{B}$ for some $j \geq 0$ (it is clear for $j = 0$). Since $up = p(u + 4v)$, we get that for all $n \in \mathbb{Z}$:

$$\widehat{B}p\widehat{B} \ni \ [\widehat{\phi}(e_n), w^jp] = (u - (n-1)w)v^{n-1}w^jp - w^jp(u - (n-1)w)v^{n-1} = (j+4)v^nw^jp.$$

So, $\Bbbk[v, v^{-1}] \cdot w^jp \subseteq \widehat{B}p\widehat{B}$. Since $u = \widehat{\phi}(e_1) \in \widehat{B}$, we have $\widehat{R}w^jp = \Bbbk[u] \cdot \Bbbk[v, v^{-1}] \cdot w^jp \subseteq \widehat{B}p\widehat{B}$. Finally, since we have seen that $v^{-1}w^jp \in \widehat{R}w^jp \subseteq \widehat{B}p\widehat{B}$, we have that

$$\widehat{B}p\widehat{B} \ni \ (\widehat{\phi}(e_1) - \widehat{\phi}(e_2)v^{-1})w^jp = w^{j+1}p.$$

By induction, $\widehat{B}p\widehat{B} = \widehat{S}p$, as desired.

From the definitions, $p \in (\ker \eta_0) \cap (\ker \eta_1)$. So

$$\widehat{B}p\widehat{B} \subseteq (\ker \eta_0) \cap (\ker \eta_1) \cap \widehat{B} = w\widehat{S} \cap (v - w)\widehat{S} = w(v - w)\widehat{S} = p\widehat{S}.$$

Combining this with the first part of the proof, we have $\widehat{B}p\widehat{B} = (\ker \eta_0) \cap (\ker \eta_1) \cap \widehat{B}$. By (3.10) and the definition of $\widehat{I}$, we have:

$$\widehat{I} = (\ker \eta_0) \cap \widehat{B} = \widehat{\phi}(\ker \widehat{\lambda}_0) = \widehat{\phi}(\ker \widehat{\lambda}_1) = (\ker \eta_1) \cap \widehat{B},$$

completing the proof. □

From Proposition 3.11 we obtain:

Theorem 3.12. *The ideal $\widehat{I}$ of $\widehat{B}$ is not finitely generated as a left or right ideal. As a result, the kernel of $\widehat{\lambda}_0$ is not finitely generated as a left or right ideal of $U(W)$.*

Proof. This argument is similar to the proof of Theorem 3.3. It suffices to show that $\widehat{I}$ is not finitely generated as a left or right ideal of $\widehat{B}$.

By way of contradiction, suppose that for some $n \in \mathbb{N}$, we have $\widehat{I} = \widehat{B}(\widehat{I}_{-n} \oplus \cdots \oplus \widehat{I}_n)$. For all $k \in \mathbb{Z}$, we have $\widehat{\phi}(e_k) \in u\widehat{S} + w\widehat{S}$. So, $\widehat{B}_k \subseteq u\widehat{S} + w\widehat{S}$ for all $k \neq 0$, and $\widehat{I}_k \subseteq u\widehat{S} + w\widehat{S}$ for all k with $|k| > n$. Note that a power of v cannot belong to $u\widehat{S} + w\widehat{S}$. So, $v^{n-3}p \notin \widehat{I}$. However, by Proposition 3.11, we get that $\widehat{I} = \widehat{S}p$ and $v^{n-3}p \in \widehat{I}$. This contradiction shows that ${}_{\widehat{B}}\widehat{I}$ is not finitely generated.

The proof that $\widehat{I}_{\widehat{B}}$ is not finitely generated is similar; we leave the details to the reader. □

Corollary 3.13. *The universal enveloping algebra $U(V)$ is neither left nor right noetherian.*

Proof. This follows directly from Theorem 3.12, since $U(W) = U(V)/(c)$. □

Remark 3.14. After the first draft of this paper was finished, we learnt of the results of Conley and Martin in [CM07]. We thank the referee for calling [CM07] to our attention. The paper considers a family of homomorphisms defined as (using their notation)

$$\pi_\gamma : U(W) \to \Bbbk[x, x^{-1}, \partial], \quad e_n \mapsto x^{n+1}\partial + (n+1)\gamma x^n.$$

Using the identification $u = x^2\partial$, $v = x$ from the discussion after Theorem 0.7, we have

$$\widehat{\lambda}_a(e_n) = (x^2\partial - (n-1)ax)x^{n-1} = x^{n+1}\partial + (1-a)(n-1)x^n.$$

The reader may verify that $\widehat{\lambda}_a(e) = x^{2(1-a)}\pi_{1-a}(e)x^{-2(1-a)}$ for all $e \in U(W)$ (where here one uses a suitable extension of $k[x, x^{-1}, \partial]$ to carry out computations). As a result,

$$\ker \widehat{\lambda}_a = \ker \pi_{1-a} \tag{3.15}$$

for all $a \in \Bbbk$.

[CM07, Theorem 1.2] shows (using (3.15)) that $\ker \widehat{\lambda}_0 = \ker \widehat{\lambda}_1 = (e_{-1}e_2 - e_0e_1 - e_1)$. Recall from Proposition 2.5 that $\ker \lambda_0$ is generated as a two-sided ideal by $g_4 := e_1e_3 - e_2^2 - e_4$. A computation gives that

$$\mathrm{ad}(e_{-1}^3)(g_4) = [e_{-1}, [e_{-1}, [e_{-1}, g_4]]] = 12(e_{-1}e_2 - e_0e_1 - e_1),$$

and it follows that

$$(g_4) = \ker \widehat{\lambda}_0 = \ker \widehat{\lambda}_1 = (e_{-1}e_2 - e_0e_1 - e_1).$$

4. The connection between the maps ϕ and ρ

For the remainder of the paper, we return to considering $U(W_+)$. The main goal of this section is to relate the map ϕ (of Definition 0.3) that played a crucial role in the proof of Theorem 3.3 to the map ρ (of Notation 0.9) that was the focus of [SW14]. We show that $\ker \phi = \ker \rho$; in fact, we have:

Theorem 4.1. *We have that* $\ker \rho = \ker \phi = \bigcap_{a \in \Bbbk} \ker \lambda_a$. *As a consequence,* $\rho(U(W_+)) \cong \phi(U(W_+))$.

Consider Notation 0.2 and the following notation for this section. Recall the definitions of X, f, τ from Notation 0.9. So, $\tau \in \mathrm{Aut}(X)$ and $\tau^* : \Bbbk(X) \to \Bbbk(X)$ is the pullback of τ. Here we take $\mu \in \mathrm{Aut}(\mathbb{P}^2)$ and $\nu \in \mathrm{Aut}(\mathbb{P}^1)$ to be morphisms of varieties, defined by

$$\mu([x:y:z]) = [x-y:y:z] \quad \text{and} \quad \nu([x:y]) = [x-y:y].$$

We denote the respective pullback morphisms by μ^* and ν^*. However, to be consistent with Lemma 1.3 (and abusing notation slightly), we still write

$$S \cong \Bbbk[x,y,z]^{\mu} \quad \text{and} \quad R \cong \Bbbk[x,y]^{\nu}.$$

We also establish the convention that $h^{\tau} := \tau^* h$ for $h \in \Bbbk(X)$, and similarly for pullback by other morphisms.

Before proving Theorem 4.1, we provide some preliminary results.

Lemma 4.2 (ψ_a, Ψ_a)**.** *For $a \in \Bbbk$, we have the following statements.*

(a) *We have a well-defined morphism $\psi_a : \mathbb{P}^1 \to X$ given by*

$$\psi_a([x:y]) = [2x^2 - 4xy - 6ay^2 : x^2 - 2xy + y^2 : -x^2 + 3xy - 2y^2 : x^2 - 4xy + 4y^2].$$

(b) $\psi_a \nu = \tau \psi_a$.

(c) *ψ_a^* extends to an algebra homomorphism $\Psi_a : \Bbbk(X)[t;\tau^*] \to \Bbbk(\mathbb{P}^1)[s;\nu^*]$, where $\Psi_a(t) = s$.*

Proof. (a,b) Both are straightforward. Part (a) is a direct computation. In Section 7.3 in the appendix, we verify that $(\psi_a\nu)^* = \nu^*\psi_a^* = \psi_a^*\tau^* = (\tau\psi_a)^*$ as maps from $\Bbbk(X) \to \Bbbk(\mathbb{P}^1)$. Thus, (b) holds.

(c) We have for all $h, \ell \in \Bbbk(X)$ and $n, m \in \mathbb{N}$ that:

$$\begin{aligned} \Psi_a(ht^n\ell t^m) &= \Psi_a(h\ell^{\tau^n}t^{n+m}) = \psi_a^*(h)\psi_a^*(\ell^{\tau^n})s^{n+m} \\ &= \psi_a^*(h)\psi_a^*(\ell)^{\nu^n}s^{n+m} = \psi_a^*(h)s^n\psi_a^*(\ell)s^m = \Psi_a(ht^n)\Psi_a(\ell t^m). \end{aligned}$$

Thus, Ψ_a is an algebra homomorphism. $\square$

Lemma 4.3 (C_a)**.** *For $a \in \Bbbk$, define the curve*

$$C_a = V(w + 6ax + (4+12a)y + (2+6a)z, xz - y^2) \subseteq X.$$

Then, ψ_a defines an isomorphism from $\mathbb{P}^1 \to C_a$.

Proof. That the image of ψ_a of Lemma 4.2(a) is contained in C_a is a straightforward verification. The inverse map to ψ_a is defined by the birational map $[w:x:y:z] \mapsto [2x+y:x+y]$; we leave the verification of the details to the reader. $\square$

Lemma 4.4 (γ). *Define a map $\gamma : R \to \Bbbk(\mathbb{P}^1)[s;\nu^*]$ as follows: if $h \in R_n = \Bbbk[x,y]_n$, let*

$$\gamma(h) = \frac{h}{x(x-y)\cdots(x-(n-1)y)}s^n.$$

Then, γ is an injective $\Bbbk$-algebra homomorphism.

Proof. Let $h \in \Bbbk[x,y]_n$ and $\ell \in \Bbbk[x,y]_m$. Then,

$$\begin{aligned}\gamma(h*\ell) &= \gamma(h\ell^{\nu^n}) = \frac{h\ell^{\nu^n}}{x(x-y)\cdots(x-(n+m-1)y)}s^{n+m}\\ &= \frac{h}{x(x-y)\cdots(x-(n-1)y)}\left(\frac{\ell}{x(x-y)\cdots(x-(m-1)y)}\right)^{\nu^n}s^{m+n}\\ &= \frac{h}{x(x-y)\cdots(x-(n-1)y)}s^n\frac{\ell}{x(x-y)\cdots(x-(m-1)y)}s^m = \quad \gamma(h)\gamma(\ell).\end{aligned}$$

So, γ is a homomorphism; injectivity is clear. □

Proposition 4.5. *Retain the notation of Lemmas 4.2 and 4.4. Let $a \in \Bbbk$. Then, $\Psi_a\rho = \gamma\lambda_a$ as maps from $U(W_+) \to \Bbbk(\mathbb{P}^1)[s;\nu^*]$, and $\ker\Psi_a\rho = \ker\lambda_a$.*

Proof. By Lemma 1.1(a), it suffices verify that the maps $\Psi_a\rho$ and $\gamma\lambda_a$ agree on e_1 and e_2. We have:

$$\Psi_a(\rho(e_1)) = \Psi_a(t) = s = \gamma(u) = \gamma(\lambda_a(e_1)).$$

We verify that

$$\psi_a^*(f) = \frac{xy-ay^2}{x^2-xy} \tag{4.6}$$

in Section 7.3 in the appendix. Thus,

$$\Psi_a(\rho(e_2)) = \psi_a^*(f)s^2 = \frac{xy-ay^2}{x^2-xy}s^2 = \gamma(uv-av^2) = \gamma\lambda_a(e_2).$$

The final statement follows from the fact that γ is injective (Lemma 4.4). □

We now prove Theorem 4.1.

Proof of Theorem 4.1. By Lemma 4.3, $\psi_a^*h = 0$ if and only if $h|_{C_a} \equiv 0$. Now, the curves C_a cover an open subset of X. (One way to see this is that, because $\bigcup_a C_a$ is dense in X and is clearly constructible, by [Har77, Exercise II.3.19(b)] it contains an open subset of X.) Thus if $h \in \Bbbk(X)$ is in the intersection $\bigcap_a \ker\psi_a^*$, then h vanishes on this open subset and so $h = 0$. So, $\bigcap_a \ker\Psi_a = \{0\}$. Thus, $\ker\rho = \bigcap_a \ker\Psi_a\rho = \bigcap_a \ker\lambda_a$, where the last equality holds by Proposition 4.5.

To show that $\ker\phi = \bigcap_a \ker\lambda_a$, define closed immersions $i_a : \mathbb{P}^1 \to \mathbb{P}^2$ for $a \in \Bbbk$ by $i_a([x:y]) = [x:y:ay]$. Then, $\mathrm{im}(i_a) = V(z-ay)$, and pullback along i_a induces the ring homomorphism

$$i_a^* : \Bbbk[x,y,z] \to \Bbbk[x,y] \quad x \mapsto x,\ y \mapsto y,\ z \mapsto ay.$$

The reader may verify that $i_a\nu = \mu i_a$, and that i_a^* is also a homomorphism from $S = \Bbbk[x,y,z]^\mu$ to $R = \Bbbk[x,y]^\nu$. In terms of u, v, w, we have

$$i_a^*(u) = u, \quad i_a^*(v) = v, \quad i_a^*(w) = av.$$

That is, $i_a^* = \eta_a|_S$, where η_a was defined in Notation 3.9. We see that $i_a^*\phi = \lambda_a$.

As with the first paragraph, the curves $V(z-ay)$ cover an open subset of $\mathbb{P}^2$: in fact, $\bigcup_a V(z-ay) \supseteq (\mathbb{P}^2 \smallsetminus V(y))$. So $\bigcap_a \ker i_a^* = \{0\}$. Thus, $\ker\phi = \bigcap_a \ker i_a^*\phi = \bigcap_a \ker\lambda_a$, completing the proof. □

5. The kernel of ϕ

In this section, we analyze the map ϕ from Definition 0.3. In particular, we verify part (c) of Theorem 0.6(c). To proceed, recall Notations 0.2, 1.2, 1.6, and 2.2.

Theorem 5.1. *The kernel of ϕ is generated as a two-sided ideal by $g := e_1e_5 - 4e_2e_4 + 3e_3^2 + 2e_6$.*

Proof. First, observe that as $e_1e_5, e_2e_4, e_3^2, e_6$ are elements of the standard basis for $U(W_+)$ (by Lemma 1.1(b)), they are linearly independent. So, we have that $g \neq 0$.

Now we verify that $\phi(g) = 0$ by using Lemma 1.3 and (1.4):

$$\begin{aligned}\phi(g) &= u(u-4w)v^4 - 4(u-w)v(u-3w)v^3 + 3(u-2w)v^2(u-2w)v^2 + 2(u-5w)v^5\\ &= x(x-4z)^{\mu}y^4 - 4(x-z)y(x-3z)^{\mu^2}y^3 + 3(x-2z)y^2(x-2z)^{\mu^3}y^2 + 2(x-5z)y^5\\ &= x(x-y-4z)y^4 - 4(x-z)y(x-2y-3z)y^3 + 3(x-2z)y^2(x-3y-2z)y^2 + 2(x-5z)y^5 \ = 0.\end{aligned}$$

We take the following notation for the rest of the proof.

Notation 5.2 (M, M', b_5, b_6, b_7, η)**.** Consider the right B-modules

$$M := uB \cap (u-w)vB \quad \text{and} \quad M' := b_5B + b_6B + b_7B, \text{ with}$$

$$\begin{aligned} b_5 &= (uv-vw)(u^3 - 6(uv-vw)u + 12u(uv-vw)),\\ b_6 &= (uv-vw)(-48(uv-3vw)v^2 - 36u(uv-2vw)v + u^4),\\ b_7 &= (uv-vw)(u^5 - 40((uv-vw)^2u - 3(uv-vw)u(uv-vw) + 3u(uv-vw)^2)).\end{aligned}$$

Moreover, take $\eta : B \to A(0)$ to be the map induced by the projection $\eta_0 : \widehat{S} \twoheadrightarrow \widehat{R} = \widehat{S}/(w)$ from Notation 3.9.

The remainder of the proof will be established through a series of lemmas.

Lemma 5.3. *We obtain that $b_5, b_6, b_7 \in uB \cap (u-w)vB$. In other words, $M' \subseteq M$.*

Proof. Let

$$r_5 := e_2(e_1^3 - 6e_2e_1 + 12e_1e_2), \tag{5.4}$$

$$r_6 := e_2(-48e_4 - 36e_1e_3 + e_1^4), \tag{5.5}$$

$$r_7 := e_2(e_1^5 - 40(e_2^2e_1 - 3e_2e_1e_2 + 3e_1e_2^2)). \tag{5.6}$$

We have as a consequence of the degree 5 relation of $U(W_+)$ in Lemma 1.1(a) that

$$r_5 = e_1(e_1^2e_2 - 3e_1e_2e_1 + 3e_2e_1^2 + 6e_2^2), \tag{5.7}$$

and as a consequence of the degree 7 relation of $U(W_+)$ in Lemma 1.1(a) that

$$r_7 = e_1(e_1^4e_2 - 5e_1^3e_2e_1 + 10e_1^2e_2e_1^2 - 10e_1e_2e_1^3 + 5e_2e_1^4 - 40e_2^3). \tag{5.8}$$

Thus $r_5, r_7 \in e_1U(W_+) \cap e_2U(W_+)$. Since $b_5 = \phi(r_5)$ and $b_7 = \phi(r_7)$, these are both in $uB \cap (uv-vw)B$.

Note that $r_6 \in e_2U(W_+)$, so $b_6 = \phi(r_6) \in (u-w)vB$. Further,

$$r_6 = e_1(-36e_2e_3 - 18e_5 + 2e_4e_1 - e_3e_1^2 + e_2e_1^3) + 12g.$$

Thus, $b_6 \in uB$ as well. □

Lemma 5.9. *Suppose that $M = M'$. Then,* $\ker\phi = (g)$ *and the theorem holds.*

Proof. Let K be the kernel of

$$\begin{aligned} \alpha : B[-1] \oplus B[-2] &\to B \\ (b, b') &\mapsto (ub + (uv - vw)b'). \end{aligned}$$

It is a standard fact that the map

$$\beta : M \to K$$

defined by $\beta(r) = (u^{-1}r, -(uv - vw)^{-1}r)$ is an isomorphism of graded right B-modules, as in the proof of Lemma 2.3. Thus, K is generated by $\beta(b_5)$, $\beta(b_6)$, and $\beta(b_7)$ by the assumption. By Proposition 7.1 in the appendix, the kernel of π_B is generated as a 2-sided ideal of $\Bbbk\langle t_1, t_2\rangle$ by a degree 5 element q_5, a degree 6 element q_6, and a degree 7 element q_7. We compute q_5 and q_7 by applying the formula from Proposition 7.1 to $\beta(b_5)$ and $\beta(b_7)$, and by using (5.4)-(5.8). Namely, take

$$\begin{aligned} \tilde{b}_1^1 &= t_1^2t_2 - 3t_1t_2t_1 + 3t_2t_1^2 + 6t_2^2, & \tilde{b}_2^1 &= -t_1^3 + 6t_2t_1 - 12t_1t_2 \\ \tilde{b}_1^2 &= t_1^4t_2 - 5t_1^3t_2t_1 + 10t_1^2t_2t_1^2 - 10t_1t_2t_1^3 + 5t_2t_1^4 - 40t_2^3 & \tilde{b}_2^2 &= -t_1^5 + 40(t_2^2t_1 - 3t_2t_1t_2 + 3t_1t_2^2). \end{aligned}$$

So, we have that

$$\begin{aligned} q_5 &= t_1\tilde{b}_1^1 + t_2\tilde{b}_2^1 = [t_1, [t_1, [t_1, t_2]]] + 6[t_2, [t_2, t_1]], \\ q_7 &= t_1\tilde{b}_1^2 + t_2\tilde{b}_2^2 = [t_1, [t_1, [t_1, [t_1, [t_1, t_2]]]]] + 40[t_2, [t_2, [t_2, t_1]]]. \end{aligned}$$

By Lemma 1.1(a), q_5 and q_7 generate the kernel of π. So, $\ker\phi = \pi(\ker \pi_B) = (\pi(q_6))$. We see immediately that $(\ker\phi)_6$ is a 1-dimensional $\Bbbk$-vector space, generated by $\pi(q_6)$. Since $g \in (\ker\phi)_6$ is nonzero, we have $g = \pi(q_6)$ up to scalar multiple. Therefore, $\ker\phi = (g)$. □

Our goal now is show that $M = M'$; we do this by comparing Hilbert series. To proceed, we show that:

Lemma 5.10. *The Hilbert series of M is $t^5(1-t)^{-2}(1-t^2)^{-1}$.*

Proof. Since $A(0) = \Bbbk \oplus uR$ we have

$$\operatorname{hilb} A(0) = 1 + t(\operatorname{hilb} R) = 1 + \frac{t}{(1-t)^2} = \frac{1-t+t^2}{(1-t)^2}.$$

On the other hand, it is well-known that $\operatorname{hilb} Q = \operatorname{hilb} \Bbbk[x, y, yz] = (1-t)^{-2}(1-t^2)^{-1}$. Since $\lambda_0 = \eta \circ \phi$, we get that $\ker\eta = \phi(\ker\lambda_0)$ (which is denoted by I in Notation 3.1). So, by Lemma 3.2(c), we get

$$\operatorname{hilb} \ker\eta = \frac{t^4}{(1-t)^2(1-t^2)}.$$

Then

$$\operatorname{hilb} B = \operatorname{hilb} A(0) + \operatorname{hilb} \ker\eta = \frac{1-t+t^3-t^4}{(1-t)^2(1-t^2)} + \frac{t^4}{(1-t)^2(1-t^2)} = \frac{1-t+t^3}{(1-t)^2(1-t^2)}.$$

Finally, we compute $\operatorname{hilb} M$ from the exact sequence

$$0 \longrightarrow M \xrightarrow{\beta} B[-1] \oplus B[-2] \xrightarrow{\alpha} B \longrightarrow \Bbbk \longrightarrow 0,$$

where α, β are as in the proof of Lemma 5.9. This gives

$$\operatorname{hilb} M = (t^2 + t - 1)(\operatorname{hilb} B) + 1 = \frac{t^5}{(1-t)^2(1-t^2)},$$

as claimed. □

We now provide results on the Hilbert series of M'.

Lemma 5.11. *We have that* $\operatorname{hilb}\eta(M') \geq t^5(1-t)^{-2}$.

Proof. Let $a_5 := \eta(b_5)$ and $a_6 := \eta(b_6)$. Then,

$$\begin{aligned} a_5 &= uvu(u^2 - 6vu + 12uv) = xy(x-2y)[(x-3y)(x-4y) - 6y(x-4y) + 12(x-3y)y] \\ &= x^2(x-y)(x-2y)y. \\ a_6 &= uvu(u^3 - 36uv^2 - 48v^3) = xy(x-2y)[(x-3y)(x-4y)(x-5y) - 36(x-3y)y^2 - 48y^3] \\ &= x^2(x-y)(x-2y)y(x-11y) \\ &= a_5(u-6v). \end{aligned}$$

Since a_5u and $a_5(u-6v)$ are in $\eta(M')$ and u and $u-6v$ span R_1, we have $a_5R_1 \subseteq \eta(M')$. We get that $\eta(M') \supseteq a_5A(0) + a_5R_1A(0)$, as $\eta(M')$ is a right $A(0)$-module and contains $a_5R_{\leq 1}$. Since $A(0) + R_1A(0) = R$, we obtain that $\eta(M') \supseteq a_5R$. Now as $\operatorname{hilb} R = (1-t)^{-2}$, we conclude that $\operatorname{hilb} \eta(M') \geq t^5(1-t)^{-2}$. □

Lemma 5.12. *We have that* $\operatorname{hilb}(M' \cap \ker \eta) \geq t^7(1-t)^{-2}(1-t^2)^{-1}$.

Proof. Again, recall that $\ker \eta = \phi(\ker \lambda_0)$, which is denoted by I in Notation 3.1. Moreover by Lemma 3.2(c), we have that $I = Qp = pQ$, where $p = v^3w - v^2w^2$. Let

$$h := (uv - vw)(u + 2v)p = (xy - yz)x(y^3z - y^2z^2).$$

Now we proceed by asserting the following:

Claim. $b_5Q + b_6Q + b_7Q \ni \; x(xy - yz)(xyz + y^2z) = (uv - vw)(u + 2v)(u + 4v)vw.$

The proof of this claim is provided in the appendix; see Claim 7.6(a).

Since $M' \cap I \supseteq M'I = b_5Qp + b_6Qp + b_7Qp$, we have

$$M' \cap I \supseteq (uv - vw)(u + 2v)(u + 4v)vwpQ = (xy - yz)x(y^3z - y^2z^2)(x + y)yzQ = h(x + y)yzQ. \tag{5.13}$$

We now show by induction that $M' \cap I \supseteq hQ_n$ for all $n \geq 0$.

Claim. $M' \cap I \supseteq hQ_n$ for $n = 0, 1, 2$.

The proof of this assertion is provided in the appendix; see Claim 7.6(b). We will prove the result for larger n by geometric arguments. The maximal graded non-irrelevant ideals of $\Bbbk[x, y, yz]$ are in bijective correspondence with $\Bbbk$-points of the weighted projective plane $\mathbb{P}(1, 1, 2)$ [Har92, Example 10.27]. We use the notation $(a : b : c)$ to denote a point of $\mathbb{P}(1, 1, 2)$. Let

$$K(n) := (x - ny)\Bbbk[x, y, yz] + (y^2 - yz)\Bbbk[x, y, yz],$$

be the graded ideal of polynomials vanishing at $(n : 1 : 1)$.

Suppose now that $M' \cap I \supseteq hQ_n$ for some $n \geq 2$. Then, $M' \cap I$ contains

$$\begin{aligned} h\left[Q_nu + Q_{n-1}(uv - vw)\right] &= h\left[(x - (n+7)y)\Bbbk[x, y, yz] + ((x - (n+6)y)y - yz)\Bbbk[x, y, yz]\right]_{n+1} \\ &= h\left[(x - (n+7)y)\Bbbk[x, y, yz] + (y^2 - yz)\Bbbk[x, y, yz]\right]_{n+1} \\ &= hK(n+7)_{n+1}. \end{aligned}$$

From (5.13), we get that $(M' \cap I)_{n+1} \ni h(xyz + y^2z)y^{n-2}$. Since $(xyz + y^2z)y^{n-2}$ does not vanish at $(n + 7 : 1 : 1)$, it is not in $hK(n + 7)_{n+1}$. Thus,

$$hK(n+7)_{n+1} + \Bbbk h(xyz + y^2z)y^{n-2} = h\Bbbk[x, y, yz]_{n+1} \subseteq M' \cap I,$$

where the equality holds as $hK(n + 7)_{n+1}$ is codimension 1 in $h\Bbbk[x, y, yz]_{n+1}$. Hence, $hQ_{n+1} \subseteq M' \cap I$.

Now by induction, we obtain that $M' \cap I \supseteq hQ$. Since $\operatorname{hilb} Q = (1-t)^{-2}(1-t^2)^{-1}$, we have

$$\operatorname{hilb}(M' \cap I) \geq \frac{t^7}{(1-t)^2(1-t^2)}.$$

□

Our final lemma is

Lemma 5.14. *We have that* $\operatorname{hilb} M = \operatorname{hilb} M' = t^5(1-t)^{-2}(1-t^2)^{-1}$. *As a result,* $M = M'$.

Proof. Combining Lemmas 5.11 and 5.12, we have

$$\mathrm{hilb}(M') \geq \frac{t^5}{(1-t)^2} + \frac{t^7}{(1-t)^2(1-t^2)} = \frac{t^5}{(1-t)^2(1-t^2)}.$$

On the other hand, by Lemmas 5.3 and 5.10 we get that

$$\mathrm{hilb}(M') \leq \frac{t^5}{(1-t)^2(1-t^2)}.$$

Thus, $\mathrm{hilb}\, M = \mathrm{hilb}\, M'$. Since $M' \subseteq M$ again by Lemma 5.3, we conclude that $M = M'$. □

Theorem 5.1 now follows from Lemmas 5.9 and 5.14. □

Remark 5.15. A shorter proof of Theorem 5.1 follows from the results of [CM07]. Recall from Notation 3.9 that we may extend ϕ to a map $\widehat{\phi} : U(W) \to \widehat{S}$, using the same formula (0.4) for $\widehat{\phi}(e_n)$ with $n \leq 0$. Then [CM07, Theorem 1.3] and (3.15), together with Theorem 4.1, give that $\ker \widehat{\phi} = (e_{-1}e_3 - 4e_0e_2 + 3e_1^2 + 2e_2)$. The reader may verify that

$$\mathrm{ad}(e_{-1}^4)(g) = [e_{-1}, [e_{-1}, [e_{-1}, [e_{-1}, g]]]] = 24(e_{-1}e_3 - 4e_0e_2 + 3e_1^2 + 2e_2).$$

Since $\widehat{\phi}(g) = 0$, we have $(g) \subseteq \ker \widehat{\phi} = (e_{-1}e_3 - 4e_0e_2 + 3e_1^2 + 2e_2) \subseteq (g)$, so all are equal.

6. A partial result on chains of two-sided ideals

It is not known whether $U(W_+)$ satisfies the ascending chain condition (ACC) on two-sided ideals; see Question 0.11. We do not answer this question here; however, we prove the partial result that the non-noetherian factor B of $U(W_+)$ does have ACC on two-sided ideals.

Recall Notations 0.2, 1.2, 1.6; in particular, Q is the subalgebra of S generated by u, v, vw. Throughout, we consider B as a subalgebra of Q. We begin by proving:

Lemma 6.1. *Let h be a nonzero, homogeneous, normal element of Q, and let $a \in \Bbbk$. Then, the Q-bimodules*

$$N := hQ/hvQ \quad \text{and} \quad M_a = hQ/h(vw - av^2)Q$$

are noetherian B-bimodules under the action induced from Q.

Proof. We remark that any normal element of Q must be in the commutative subalgebra $\Bbbk[v, vw]$, and thus, must commute with v and vw. In particular, $vQN = 0$ and $(vw - av^2)QM_a = 0 = M_a(vw - av^2)Q$.

Let $\theta : Q \to Q/vQ$ be the canonical projection. (Note that $vw \notin \ker \theta$.) Since $u(vw) - (vw)u = 2v^2w$ is contained in $\ker \theta$, the image Q/vQ is commutative. It is easy to see that $Q/vQ \cong \Bbbk[s,t]$ under the identification $s = \theta(u)$, $t = \theta(vw) = \theta(uv - vw)$. Note that $s = \theta(\phi(e_1))$ and $t = \theta(\phi(e_2))$ are in B. So, $\theta(B) = Q/vQ$. Thus, a left B-submodule of hQ/hvQ is simply an ideal of $\Bbbk[s,t]$. So, hQ/hvQ is noetherian as a left B-module. As chains of B-bimodules are also chains of left B-modules, hQ/hvQ is also a noetherian B-bimodule.

Now define an algebra homomorphism $\delta : Q \to R$ by $\delta(u) = u$, $\delta(v) = v$, and $\delta(vw) = av^2$. (Note that $\delta = \eta_a|_Q$ from Notation 3.9.) It is easy to see that $\ker \delta = (vw - av^2)Q$ and that δ is surjective. Note also that $\delta(\phi(e_1)) = u$ and $\delta(\phi(e_2)) = uv - av^2$. Thus, $\delta(B) = A(a)$ as subalgebras of R. If $a \neq 0, 1$, then by Proposition 2.1, $A(a) \supseteq R_{\geq 4}$ is noetherian, and R is a finitely generated right $A(a)$-module. If $a = 0$, then $R = A(0) + vA(0)$ is again a finitely generated right $A(0)$-module, and $A(0)$ is noetherian. Thus for $a \neq 1$, M_a is also a finitely generated right $A(a)$-module. So, M_a is noetherian as a right B-module, let alone a B-bimodule.

If $a = 1$ then we have, similarly, that $\delta(B) = A(1)$ is noetherian, and that $R = A(1) + A(1)v$ is a finitely generated left $A(1)$-module. It follows that M_a is a finitely generated left $A(a)$-module. So, M_a is noetherian as a left B-module, and again as a B-bimodule. □

We now use geometric arguments to show:

Proposition 6.2. *Suppose that $\Bbbk$ is algebraically closed, and let $K \subseteq Q$ be a nonzero graded ideal. Then, Q/K is a noetherian B-bimodule.*

Proof. Let T be the commutative ring $\Bbbk[x, y, yz]$. We consider K as a subset of T, since (via Lemma 1.3) $Q = T^{\mu}$ and T have the same underlying vector space. For all $n, m \in \mathbb{N}$, we have

$$K_{n+m} \supseteq K_n Q_m = K_n (T_m)^{\mu^n} = K_n T_m, \tag{6.3}$$

and so K is also an ideal of T. Further,

$$K_{n+m} \supseteq Q_m K_n = T_m (K_n)^{\mu^m}. \tag{6.4}$$

If T were generated in degree 1, one could obtain directly from (6.3), (6.4) that K_n is μ-invariant for $n \gg 0$ (or see [AS95, Lemma 4.4]). A similar statement holds in our case; however, a proof would take us too far afield so we work more directly with the graded pieces of K.

Choose n_0 so that $K_{n_0} \neq 0$. For all $n \geq n_0$, let $h_n \neq 0$ be a greatest common divisor of K_n, considered as a subset of T_n. By (6.3), $h_{n+1} \mid h_n x, h_n y$. Since x, y have no common divisor, we have $h_{n+1} \mid h_n$ for all $n \geq n_0$. This chain of divisors must stabilize, and thus there is $n_1 \geq n_0$ so that $h_{n+1} h_n^{-1} \in \Bbbk$ for $n \geq n_1$. Let $h := h_{n_1}$.

By (6.4), $h \mid \mu^m(h)$ for all $m \in \mathbb{N}$, so h is an eigenvector of μ. Thus, h is normal in Q. Since $h \mid f$ for all $f \in K$, we can write $K = hJ$ for some $J \subseteq Q$. Since h is normal, J is again an ideal of Q. So, (6.3), (6.4) apply to J.

Since $h \in \Bbbk[v, vw]$ and $\Bbbk$ is algebraically closed, we have

$$h = (vw - a_1 v^2) \cdots (vw - a_n v^2) v^k$$

for some $n, k \in \mathbb{N}$ and $a_1, \ldots, a_n \in \Bbbk$. Applying Lemma 6.1 repeatedly, we obtain that Q/hQ is a noetherian B-bimodule.

From the exact sequence

$$0 \to hQ/hJ \to Q/K \to Q/hQ \to 0,$$

it suffices to prove that hQ/hJ is a noetherian B-bimodule. We make a geometric argument to do so.

Graded ideals of T correspond to subschemes of the weighted projective plane $\mathbb{P}(1,1,2)$. Note that μ acts on $\mathbb{P}(1,1,2)$ by $\mu(a : b : c) = (a - b : b : c)$.

Let Y_n be the subset of $\mathbb{P}(1,1,2)$ defined by the vanishing of the polynomials in J_n, considered now as a subset of T. By the definition of h, for $n \geq n_1$ the polynomials in J_n have no nontrivial common factor, and so $\dim Y_n \leq 0$. By (6.3), (6.4), we have

$$Y_{n+1} \subseteq Y_n \cap \mu(Y_n)$$

for $n \geq n_1$. It follows that there exists $n_2 \geq n_1$ so that

$$Y_{n+1} = Y_n = \mu(Y_n) \tag{6.5}$$

for $n \geq n_2$. Let $Y := Y_{n_2}$. Since μ-orbits in $\mathbb{P}(1,1,2)$ are either infinite or trivial, each point of Y is μ-invariant. Note that Y is the subset of $\mathbb{P}(1,1,2)$ defined by J, considered as an ideal of T.

Let P be an associated prime of J. Since J is graded, P is graded. By using the Nullstellensatz, with the fact that $\dim Y \leq 0$, we get that either $P = T_+$, or P defines some point $(a : b : c) \in Y$. In the first case, certainly $y \in P$. In the second case, $(a : b : c) = \mu(a : b : c) = (a - b : b : c)$ and so $b = 0$. Again, $y \in P$.

The radical $\sqrt{J}$ is the intersection of the associated primes of J. Since y is contained in all associated primes, $y \in \sqrt{J}$. Thus, there is some n so that $y^n = v^n \in J$. So, hQ/hJ is a factor of $hQ/hv^n Q$. Applying Lemma 6.1 again, we see that hQ/hJ is a noetherian B-bimodule, as desired. $\square$

We now prove Proposition 0.12. In fact, we show:

Proposition 6.6. *The ring Q is noetherian as a B-bimodule. As a consequence, B satisfies ACC on two-sided ideals.*

Proof. Let $\Bbbk'$ be an algebraic closure of $\Bbbk$. If $Q \otimes_{\Bbbk} \Bbbk'$ were a noetherian bimodule over $B \otimes_{\Bbbk} \Bbbk'$, then Q would be a noetherian B-bimodule; this holds as $\Bbbk'$ is faithfully flat over $\Bbbk$ [GW04, Exercise 17T]. So it suffices to prove the result in the case that $\Bbbk$ is algebraically closed. By standard arguments, it is sufficient to show that Q satisfies ACC on *graded* B-subbimodules, or equivalently, that any nonzero graded B-subbimodule of Q is finitely generated.

Let K be a nonzero graded B-subbimodule of Q. Since $B \supseteq Qp = pQ$ by Lemma 3.2(c), we have that $K = BKB \supseteq QpKpQ$. Since Q is noetherian, there is a finite dimensional graded vector space $V \subseteq K$ with $QpKpQ = QpVpQ$.

By Proposition 6.2, the B-bimodule $Q/QpVpQ$ is noetherian. The B-subbimodule $K/QpVpQ$ of $Q/QpVpQ$ is thus finitely generated. So, there is a finite-dimensional vector space $W \subseteq K$ so that $K = BWB + QpVpQ \subseteq BWB + BVB$. As $V, W \subseteq K$, certainly $K \supseteq BWB + BVB$. Thus, K is finitely generated by $V + W$, as needed. $\square$

7. Appendix

We first give a general result from ring theory to which we were not able to find a reference; it is the converse to [Rog, Lemma 2.11]. We then finish by presenting Maple and Macaulay2 routines and proofs of computational claims asserted above.

7.1. **A result in ring theory.** Consider the following setting. Let $T = \Bbbk\langle t_1, \dots, t_n\rangle$ be the free algebra. Set $\deg t_i = d_i \in \mathbb{Z}_{\geq 1}$, and grade T by the induced grading. Suppose that $\pi : T \to A$ is a surjective homomorphism of graded algebras, and let $a_i = \pi(t_i)$. By definition, the a_i generate A as an algebra. Let $J = \ker \pi$. Consider the map

$$\alpha : A[-d_1] \oplus \cdots \oplus A[-d_n] \xrightarrow{(a_1,\dots,a_n)} A$$

that sends $(r_1, \dots, r_n) \mapsto \sum_{i=1}^n a_i r_i$. Note α is a homomorphism of graded right A-modules, and set $K = \ker \alpha$. Let $b^1, \dots, b^m$ be homogeneous elements of K, where $b^j = (b^j_1, \dots, b^j_n) \in A[-d_1] \oplus \cdots \oplus A[-d_n]$. For all $1 \leq i \leq n, 1 \leq j \leq m$, choose homogenous elements $\widetilde{b}^j_i \in T$ so that $\pi(\widetilde{b}^j_i) = b^j_i$. Let $q_j = \sum_{i=1}^n t_i \widetilde{b}^j_i$. (Note that the q_i are homogeneous; in fact, $\deg q_j = \deg b^j$.)

Proposition 7.1. *Retain the notation above. If $\{b^1, \dots, b^m\}$ generate K as a right A-module, then $\{q_1, \dots, q_m\}$ generate J as an ideal of T.*

Proof. Let J' be the ideal of T generated by $q_1, \dots, q_m$. Since $\pi(q_j) = \sum_i \pi(t_i)\pi(\widetilde{b}^j_i) = \sum_i a_i b^j_i = \alpha(b^j) = 0$, we get that $J' \subseteq J$.

We prove by induction that $J'_k = J_k$ for all $k \in \mathbb{N}$. Certainly $J'_0 = J_0 = 0$. Assume that we have shown that $J'_{<k} = J_{<k}$, and let $h \in J_k$. Because T is generated by $t_1, \dots, t_n$, there are homogeneous elements $f_1, \dots, f_n \in T$, with $\deg f_i = k - d_i$, so that $h = \sum_i t_i f_i$. Then,

$$0 = \pi(h) = \sum_{i=1}^n a_i \pi(f_i) = \alpha(\pi(f_1), \dots, \pi(f_n)).$$

Since the b^j generate $K = \ker \alpha$, there are homogeneous elements $r_1, \dots, r_m \in A$ with $(\pi(f_1), \dots, \pi(f_n)) = \sum_{j=1}^m b^j r_j$. Let $\widetilde{r}_1, \dots, \widetilde{r}_m$ be homogeneous lifts of $r_1, \dots, r_m$. Then for each i we have

$$\pi(f_i) = \sum_j b^j_i r_j = \sum_j \pi(\widetilde{b}^j_i \widetilde{r}_j).$$

So, $f_i - \sum_j \widetilde{b}_i^j \widetilde{r}_j \in J = \ker \pi$. Since $\deg f_i = k - d_i < k$, each $f_i - \sum_j \widetilde{b}_i^j \widetilde{r}_j \in J'$. Thus J' contains

$$\sum_i t_i f_i - \sum_i t_i \left(\sum_j \widetilde{b}_i^j \widetilde{r}_j \right) = h - \sum_j \left(\sum_i t_i \widetilde{b}_i^j \right) \widetilde{r}_j = h - \sum_j q_j \widetilde{r}_j.$$

As $\sum_i t_i \widetilde{b}_i^j = q_j \in J'$ by definition, we have $\sum_j q_j \widetilde{r}_j \in J'$. Therefore, $h \in J'_k$. □

7.2. **Proof of assertions: Maple routines.** We begin with the following Maple routine.

Routine 7.2. A Maple routine to compute the kernel of λ_a at a specific degree n is presented as follows.

Recall from Lemma 1.1(b) that a $\mathbb{k}$-vector space basis of $U(W_+)_n$ is given by partitions of n. Moreover, we employ Lemma 1.3 and (1.4) to input a function $f(i,j) = \lambda_a(e_i)^{\mu^j}$, considered as an element of $\mathbb{k}[x,y]$.

```
with(combinat,partition):   with(LinearAlgebra):
# Choose value of n
n:=1;
N:=partition(n):   f:=(i,j)->((x-j*y)-(i-1)*a*y)*y^(i-1):
```

Given a partition $d := (n_1, \ldots, n_k)$ of n, we create a list of double indexed entries $m = (m[i_1, j_1], \ldots, m[i_k, j_k])$. Here, $i_\ell = n_\ell$, and $j_1 = 0$ with $j_\ell = j_{\ell-1} + n_{\ell-1}$ for $\ell \geq 2$. Then, $\lambda_a(e_{n_1} \cdots e_{n_k}) = m[i_1, j_1] \cdots m[i_k, j_k]$, denoted by P. (Here, P in list form, which we put in matrix form later for multiplication. The k-loop enables us to form the product of elements $m[i_*, j_*]$.)

```
P:=[]:
for d from 1 to nops(N) do        M:=[]:                     j[1]:=0:
for l from 1 to nops(N[d]) do     j[l+1]:=j[l]+ N[d][l]:     M:=[op(M),f(N[d][l],j[l])]:    S[0]:=1:
for k from 1 to nops(M) do        S[k]:=S[k-1]*M[k]:
end do:   end do:
P:=[op(P),expand(S[nops(M)])]:
end do:
```

Next, we define an arbitrary element of $\lambda_a(U(W_+)_n)$, namely $p := \sum_{i=1}^k b_i \lambda_a(e_{n_i})$.

```
B:=[];
for i from 1 to nops(N) do        B:=[op(B),b[i]]:           end do:
Bvec:=convert(B,Matrix):          Pvec:=convert(P,Matrix):   q:=Multiply(Bvec,Transpose(Pvec)):
p:=expand(q[1][1]):
```

Then, we set the coefficients of p equal to 0 and solve for the b_i. We rule out the case when $a = 0, 1$.

```
Coeffs:=[coeffs(collect(p,[x,y], 'distributed'),[x,y])]:
solve([op(Coeffs),a<>0,a<>1]);
```

Note that the number of free b_i equals the $\mathbb{k}$-vector space dimension of $(\ker \lambda_a)_n$.

We continue by verifying the claim from the proof of Lemma 2.6.

Claim 7.3. Retain the notation from Section 2, especially in Lemma 2.6. We have that $J_5 A(a)_2 \not\subseteq J_6 A(a)_1$.

Proof. Nonzero elements in J_5 arise as elements of $(u - av)vA(a)_3$ that are divisible by u on the left. We obtain that

$$\begin{aligned}(u-av)vA(a)_3 &= \mathbb{k}[(uv - av^2)(u^3)] \oplus \mathbb{k}[(uv - av^2)(u(u-av)v)] \oplus \mathbb{k}[(uv - av^2)((u-2av)v^2)] \\ &= \mathbb{k}[r_1] \oplus \ \mathbb{k}[r_2] \oplus \ \mathbb{k}[r_3]\end{aligned}$$

where

$$\begin{aligned} r_1 &:= u^4v - (3+a)u^3v^2 + (6+6a)u^2v^3 - (6+18a)uv^4 + 24av^5,\\ r_2 &:= u^3v^2 - (2+2a)u^2v^3 + (2+5a+a^2)uv^4 - (6a+2a^2)v^5,\\ r_3 &:= u^2v^3 - (1+3a)uv^4 + (2a+2a^2)v^5. \end{aligned}$$

We see this as $v^ku = uv^k - kv^{k+1}$ for all $k \geq 1$, $vu^2 = u^2v - 2uv^2 + 2v^3$, $v^2u^2 = u^2v^2 - 4uv^3 + 6v^4$, $vu^3 = u^3v - 3u^2v^2 + 6uv^3 - 6v^4$, and $v^2u^3 = u^3v^2 - 6u^2v^3 + 18uv^4 - 24v^5$ in R. Eliminating the v^5 term of r_1, r_2, r_3, we get that J_5 is generated by

$$\begin{aligned} s_1 &:= (3+a)r_1 + 12r_2,\\ s_2 &:= (1+a)r_1 - 12r_3,\\ s_3 &:= (1+a)r_2 + (3+a)r_3. \end{aligned}$$

By way of contradiction, suppose that $J_5A(a)_2 \subseteq J_6A(a)_1$. Recall that $J \subseteq L$, where $L := uR \cap (u-av)vR$. Further, $J_6 = L_6$, and $L = rR$ for

$$r = u(uv + (1-a)v^2) = (uv - av^2)(u + 2v).$$

So, $s_i = r(c_{i1}u^2 + c_{i2}uv + c_{i3}v^2) \in J_5 \subseteq rR_2$, for some $c_{ij} \in \Bbbk$. We produce these coefficients c_{ij} below.

```
r1:=x*(x-y)*(x-2*y)*(x-3*y)*y-(3+a)*x*(x-y)*(x-2*y)*y^2+(6+6*a)*x*(x-y)*y^3-(6+18*a)*x*y^4+24*a*y^5:
r2:=x*(x-y)*(x-2*y)*y^2-(2+2*a)*x*(x-y)*y^3+(2+5*a+a^2)*x*y^4-(6*a+2*a^2)*y^5:
r3:=x*(x-y)*y^3-(1+3*a)*x*y^4+(2*a+2*a^2)*y^5:
s1:=(3+a)*r1+12*r2:                s2:=(1+a)*r1-12*r3:              s3:=(1+a)*r2+(3+a)*r3:
r:=x*((x-y)*y+(1-a)*y^2):
eq1:=s1 - r*(c11*(x-3*y)*(x-4*y)+c12*(x-3*y)*y+c13*y^2):
eq2:=s2 - r*(c21*(x-3*y)*(x-4*y)+c22*(x-3*y)*y+c23*y^2):
eq3:=s3 - r*(c31*(x-3*y)*(x-4*y)+c32*(x-3*y)*y+c33*y^2):
Coeffs1:=[coeffs(collect(eq1,[x,y], 'distributed'),[x,y])]:
Coeffs2:=[coeffs(collect(eq2,[x,y], 'distributed'),[x,y])]:
Coeffs3:=[coeffs(collect(eq3,[x,y], 'distributed'),[x,y])]:
solve(Coeffs1);                solve(Coeffs2);                solve(Coeffs3);
>            {a = a, c11 = 3 + a, c12 = 6 - 2 a, c13 = -4 a}
>            {a = a, c21 = 1 + a, c22 = -2 - 2 a, c23 = -4 + 8 a}
                                                      2
>            {a = a, c31 = 0, c32 = 1 + a, c33 = 1 - 2 a - a }
```

Therefore,

$$\begin{aligned} s_1 &= r((3+a)u^2 + (6-2a)uv - 4av^2),\\ s_2 &= r((1+a)u^2 - (2+2a)uv - (4-8a)v^2),\\ s_3 &= r((1+a)uv + (1-2a-a^2)v^2). \end{aligned}$$

Now by assumption, for $i = 1, 2, 3$ we have $s_i(u-av)v = w_iu$ for some $w_i \in J_6$. Take an arbitrary element of $J_6 = L_6 = rR_3$, namely $r(d_{i1}u^3 + d_{i2}u^2v + d_{i3}uv^2 + d_{i4}v^3)$ for $d_{ij} \in \Bbbk$. Then, for some $\alpha_i \in \Bbbk$,

$$p_i := s_i(u-av)v = \alpha_i r(d_{i1}u^4 + d_{i2}u^2vu + d_{i3}uv^2u + d_{i4}v^3u). \tag{7.4}$$

Continuing with the code we enter:

```
s1:=r*((3+a)*(x-3*y)*(x-4*y)+(6-2*a)*(x-3*y)*y-4*a*y^2):
s2:=r*((1+a)*(x-3*y)*(x-4*y)-(2+2*a)*(x-3*y)*y-(4-8*a)*y^2):
s3:=r*((1+a)*(x-3*y)*y+(1-2*a-a^2)*y^2):
p1:=s1*(x-(5+a)*y)*y:              p2:=s2*(x-(5+a)*y)*y:         p3:=s3*(x-(5+a)*y)*y:
Eq1:=p1 - alpha1*r*(d11*(x-3*y)*(x-4*y)*(x-5*y)*(x-6*y) + d12*(x-3*y)*(x-4*y)*y*(x-6*y)
                  +d13*(x-3*y)*y^2*(x-6*y) + d14*y^3*(x-6*y)):
Eq2:=p2 - alpha2*r*(d21*(x-3*y)*(x-4*y)*(x-5*y)*(x-6*y) + d22*(x-3*y)*(x-4*y)*y*(x-6*y)
                  +d23*(x-3*y)*y^2*(x-6*y) + d24*y^3*(x-6*y)):
Eq3:=p3 - alpha3*r*(d31*(x-3*y)*(x-4*y)*(x-5*y)*(x-6*y) + d32*(x-3*y)*(x-4*y)*y*(x-6*y)
```

```
                 +d33*(x-3*y)*y^2*(x-6*y) + d34*y^3*(x-6*y)):
CCoeffs1:=[coeffs(collect(Eq1,[x,y], 'distributed'),[x,y])]:
CCoeffs2:=[coeffs(collect(Eq2,[x,y], 'distributed'),[x,y])]:
CCoeffs3:=[coeffs(collect(Eq3,[x,y], 'distributed'),[x,y])]:
L1:=solve(CCoeffs1):              L2:=solve(CCoeffs2):          L3:=solve(CCoeffs3):
for i from 1 to nops([L1]) do     print(L1[i][1]);              end do;
>             a = 9,    a = 1
for i from 1 to nops([L2]) do     print(L2[i][1]);              end do;
>             a = 1,    a = 1/2
for i from 1 to nops([L3]) do     print(L3[i][1]);              end do;
                                                  2
>             a = 1,    a = RootOf(-2 - 3 _Z + _Z ) - 1
```

So in order for (7.4) to hold for $i = 1, 2, 3$, we must have $a = 1$. This yields a contradiction as desired. □

We now verify the claim from the proof of Proposition 2.8.

Claim 7.5. Retain the notation from Section 2, especially in Proposition 2.8. We have that h_2, h_3, e_1h_1, h_1e_1 are $\Bbbk$-linearly independent and that

$$h_4 = 2a(2a+1)h_2 - h_3 - (6+4a)e_1h_1 + (2+4a)h_1e_1, \quad h_5 = 4a^2h_2 - h_3 - (4+4a)e_1h_1 + (4a)h_1e_1.$$

Proof. This is established simply by considering the following linear combination

$$c_1h_2 + c_2h_3 + c_3h_4 + c_4h_5 + c_5e_1h_1 + c_6h_1e_1,$$

setting the coefficients of the basis elements of $U(W_+)_6$ equal to 0, and solving for $c_1, \dots, c_6$. By Lemma 1.1(a), the basis elements of $U(W_+)_6$ are

$$e_1^6,\ e_1^4e_2,\ e_1^2e_2^2,\ e_2^3,\ e_1^3e_3,\ e_1e_2e_3,\ e_3^2,\ e_1^2e_4,\ e_2e_4,\ e_1e_5,\ e_6.$$

So, we establish the claim via the following Maple routine:

```
with(LinearAlgebra):
M:=Matrix([
[0,  0,  0,  0,  0,      0,            3,     0,   -4,              1,                    2],
[0,  0, -4, -4,  4,      0, 20*a^2+14*a-7,     0,    0,  -16*a^2-18*a-5,  16*a^3+36*a^2+16*a-2],
[0,  0,  0,  4,  0,     -4,        7-4*a,     0,    0,          4*a+1,          -4*a^2- 4*a+2],
[0,  0,  0,  4,  0,      0,       7-14*a,    -4,    0,         14*a+5,         -12*a^2-16*a+2],
[0,  0,  1,  0, -1,   -2*a,            0, 2*a+1,    0,        -a^2-a,                      0],
[0,  0,  1,  0, -1, -2*a-2,          2*a, 2*a+3,  4*a,     -a^2-7*a-2,             4*a^2+4*a]
]);
P:=Matrix([
[c1,  0,  0,  0,  0,  0],
[ 0, c2,  0,  0,  0,  0],
[ 0,  0, c3,  0,  0,  0],
[ 0,  0,  0, c4,  0,  0],
[ 0,  0,  0,  0, c5,  0],
[ 0,  0,  0,  0,  0, c6]
]);
B:=Multiply(P,M);
for i from 1 to 11 do          L[i]:=B[1,i]+B[2,i]+B[3,i]+B[4,i]+B[5,i]+B[6,i]:          end do:
V:=solve([L[1],L[2],L[3],L[4],L[5],L[6],L[7],L[8],L[9],L[10],L[11]],[c1,c2,c3,c4,c5,c6]);
>[[c1 = -2 (c3 + 2 c3 a + 2 c4 a) a,    c2 = c3 + c4,    c3 = c3,    c4 = c4,
   c5 = 6 c3 + 4 c4 + 4 c3 a + 4 c4 a,  c6 = -2 c3 - 4 c3 a - 4 c4 a]]
eval(V,[c3=1,c4=0]);
>  [[c1 = -2 (2 a + 1) a, c2 = 1, 1 = 1, 0 = 0, c5 = 6 + 4 a, c6 = -2 - 4 a]]
eval(V,[c3=0,c4=1]);
```

```
>                     2
        [[c1 = -4 a , c2 = 1, 0 = 0, 1 = 1, c5 = 4 + 4 a, c6 = -4 a]]
```

□

Now verify the claims from the proof of Lemma 5.12.

Claim 7.6. Retain the notation from Lemma 5.12. We have the following statements.

(a) $b_5Q + b_6Q + b_7Q \ni\ x(xy - yz)(xyz + y^2z) = (uv - vw)(u + 2v)(u + 4v)vw$.

(b) $(M' \cap \ker \eta) \supseteq hQ_i$ for $i \leq 2$, where

$$h = (uv - vw)(u + 2v)(v^3w - v^2w^2) = (xy - yz)x(y^3z - y^2z^2).$$

Proof. (a) We see that $-\frac{1}{6}b_5u + b_5v + \frac{1}{6}b_6 = (uv - vw)(u + 2v)(u + 4v)vw$ by using Lemma 1.3 and (1.4):

```
b5:=(x*y-y*z)*((x-2*y)*(x-3*y)*(x-4*y)-6*((x-2*y)*y-y*z)*(x-4*y)+12*(x-2*y)*((x-3*y)*y-y*z)):
b6:=(x*y-y*z)*(-48*((x-2*y)*y-3*y*z)*y^2-36*(x-2*y)*((x-3*y)*y-2*y*z)*y+(x-2*y)*(x-3*y)*(x-4*y)*(x-5*y)):
r:=x*(x*y-y*z)*(x*y*z+y^2*z):
p:=c1*b5*(x-5*y)+c2*b5*y+c3*b6 - r:
Coeffs:=[coeffs(collect(p,[x,y,z], 'distributed'),[x,y,z])]:
solve(Coeffs);
>                        {c1 = -1/6, c2 = 1, c3 = 1/6}
```

(b) It is easy to see that $\eta(h) = 0$, so it suffices to show that hQ_0, hQ_1, hQ_2 are in $M' := b_5B + b_6B + b_7B$. Recall that Q is the subalgebra of S generated by u, v, vw, and B is the subalgebra of S generated by $u, uv - vw$. Since $\deg(h) = 7$,

$$\begin{aligned} hQ_0 &= \{c_1h \mid c_1 \in \Bbbk\},\\ hQ_1 &= \{c_2hu + c_3hv \mid c_i \in \Bbbk\},\\ hQ_2 &= \{c_4hu^2 + c_5huv + c_6hv^2 + c_7hvw \mid c_i \in \Bbbk\}, \end{aligned}$$

and moreover,

$$\begin{aligned} M'_7 &= \{d_1b_5u^2 + d_2b_5(uv - vw) + d_3b_6u + d_4b_7 \mid d_i \in \Bbbk\},\\ M'_8 &= \{d_5b_5u^3 + d_6b_5u(uv - vw) + d_7b_5(uv - vw)u + d_8b_6u^2 + d_9b_6(uv - vw) + d_{10}b_7u \mid d_i \in \Bbbk\},\\ M'_9 &= \{d_{11}b_5u^4 + d_{12}b_5u^2(uv - vw) + d_{13}b_5u(uv - vw)u + d_{14}b_5(uv - vw)u^2 + d_{15}b_5(uv - vw)^2\\ &\qquad + d_{16}b_6u^3 + d_{17}b_6u(uv - vw) + d_{18}b_6(uv - vw)u + d_{19}b_7u^2 + d_{20}b_7(uv - vw) \mid d_i \in \Bbbk\}, \end{aligned}$$

Continuing with the code in part (a), we enter:

```
b7:=(x*y-y*z)*((x-2*y)*(x-3*y)*(x-4*y)*(x-5*y)*(x-6*y)-40*(((x-2*y)*y-y*z)*((x-4*y)*y-y*z)*(x-6*y)
            -3*((x-2*y)*y-y*z)*(x-4*y)*((x-5*y)*y-y*z)+3*(x-2*y)*((x-3*y)*y-y*z)*((x-5*y)*y-y*z))):
h:=(x*y-y*z)*x*(y^3*z-y^2*z^2):
hQ0:=c1*h:
hQ1:=c2*h*(x-7*y)+c3*h*y:
hQ2:=c4*h*(x-7*y)*(x-8*y)+c5*h*(x-7*y)*y+c6*h*y^2+c7*h*y*z:
m7:=d1*b5*(x-5*y)*(x-6*y)+d2*b5*((x-5*y)*y-y*z)+d3*b6*(x-6*y)+d4*b7:
m8:=d5*b5*(x-5*y)*(x-6*y)*(x-7*y)+d6*b5*(x-5*y)*((x-6*y)*y-y*z)+d7*b5*((x-5*y)*y-y*z)*(x-7*y)
                              +d8*b6*(x-6*y)*(x-7*y)+d9*b6*((x-6*y)*y-y*z)+d10*b7*(x-7*y):
m9:=d11*b5*(x-5*y)*(x-6*y)*(x-7*y)*(x-8*y)+d12*b5*(x-5*y)*(x-6*y)*((x-7*y)*y-y*z)
                              +d13*b5*(x-5*y)*((x-6*y)*y-y*z)*(x-8*y)+d14*b5*((x-5*y)*y-y*z)*(x-7*y)*(x-8*y)
                              +d15*b5*((x-5*y)*y-y*z)*((x-7*y)*y-y*z)+d16*b6*(x-6*y)*(x-7*y)*(x-8*y)
                              +d17*b6*(x-6*y)*((x-7*y)*y-y*z)+d18*b6*((x-6*y)*y-y*z)*(x-8*y)
                              +d19*b7*(x-7*y)*(x-8*y)+d20*b7*((x-7*y)*y-y*z):
p7:=m7 - hQ0:            p8:=m8 - hQ1:            p9:=m9 - hQ2:
Coeffs7:=[coeffs(collect(p7,[x,y,z], 'distributed'),[x,y,z])]:
Coeffs8:=[coeffs(collect(p8,[x,y,z], 'distributed'),[x,y,z])]:
Coeffs9:=[coeffs(collect(p9,[x,y,z], 'distributed'),[x,y,z])]:
solve(Coeffs7,[d1,d2,d3,d4]);
```

```
                                 c1         c1           c1         c1
>                              [[d1 = - ----, d2 = ----, d3 = - ----, d4 = ----]]
                                 24         4            48         16
solve(Coeffs8,[d5,d6,d7,d8,d9,d10]);
              c2     c3         c3         c2     c3           c2    c3         c3          c2     c3
>      [[d5 = - ---- - ----, d6 = ----, d7 = ---- + ----, d8 = - ---- + ---, d9 = ----, d10 = ---- + ----]]
              24     48         24         4      16           48    192        48          16     64
solve(Coeffs9,[d11,d12,d13,d14,d15,d16,d17,d18,d19,d20]);
                c4    c6     c5     c7                               9 c4    c6    25 c5    11 c7
>[[d11 = 8 d16 + ---- + --- - ---- - ----,  [...],   d20 = -108 d16 - ---- - ---- + ----- + -----]]
                8     144    18     18                               4      24     48       24
```

Thus, all arbitrary elements of hQ_0, hQ_1, hQ_2 are contained, respectively, in M'_7, M'_8, M'_9, as desired. $\square$

7.3. **Proof of assertions: Macaulay2 routines.** The following Macaulay2 code verifies Lemma 4.2(b) and (4.6); see lines o7-o10 and line o13, respectively.

```
Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases, PrimaryDecomposition,
ReesAlgebra, TangentCone
i1 : ringX=QQ[w,x,y,z]/ideal(x*z-y^2);
i2 : taustar=map(ringX,ringX,{w-2*x+2*z,z,-y-2*z,x+4*y+4*z});
i3 : ringP1a=QQ[x,y,a];
i4 : mustar=map(ringP1a, ringP1a, {x-y,y,a});
i5 : psistar=map(ringP1a, ringX, {2*x^2-4*x*y-6*a*y^2,x^2-2*x*y+y^2,-x^2+3*x*y-2*y^2,x^2-4*x*y+4*y^2});
i6 : use ringX;
i7 : mustar(psistar(w))==psistar(taustar(w))        o7 = true
i8 : mustar(psistar(x))==psistar(taustar(x))        o8 = true
i9 : mustar(psistar(y))==psistar(taustar(y))        o9 = true
i10 : mustar(psistar(z))==psistar(taustar(z))       o10 = true
i11 : num=w+12*x+22*y+8*z;
i12 : den=12*x+6*y;
                                                               2
                                                          - y a + x*y
i13 : psistar(num)/psistar(den)                     o13 = -----------          o13 : frac(ringP1a)
                                                               2
                                                            x  - x*y
```

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